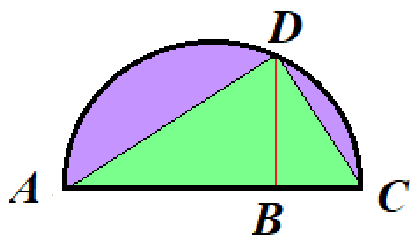
The mean proportional construction

Given two positive real number u and v, the ***mean proportional*** of u and v is defined by the equation $x^2 = uv$. A basic construction in Euclidean geometry gives a geometrical way of constructing a mean proportional by straightedge and compass constructions. If we take $v = 1$ then the construction specializes to yield a geometric construction for the square root of u. The starting point is to construct segments $[AB]$ and $[BC]$ such that $A*B*C$ with $|AB| = u$ and $|BC| = v$.



In a plane containing the line AC, construct a semicircle whose diameter equals $|AC| = u + v$ and also construct a line in this plane which is perpendicular to AC at B. The point B lies inside the circle, so the perpendicular line must meet the semicircle at some point D.

CLAIM: $x = |BD|$ *is the mean proportional of* u *and* v.

PROOF: The angle DAC is a right angle because it is inscribed in a semicircle. Thererfore we have right triangle similarities

$$\Delta DAC \sim \Delta BAD \quad \text{and} \quad \Delta DAC \sim \Delta BDC$$

(by angle – angle similarity) which yield the proportionality equation $|BD|/|BA| = |BC|/|BD|$. If we plug in the known distances between these points the equation reduces to $x/u = v/x$ or equivalently $x^2 = uv$.

Note. Similarity statements of the form $\Delta ABC \sim \Delta DEF$ implicitly assume that the vertices are ordered compatibly (so the first vertex on the left corresponds to the first vertex on the right, etc.). For example, the similarity statements $\Delta ABC \sim \Delta DEF$ and $\Delta ABC \sim \Delta DFE$ are ***not*** logically equivalent; the same sort of considerations arise in the definition of congruent triangles.