On the other hand, he found r could not be represented as a quotient $r = a/b$ of integers, for $(a/b)^2 = 2$ would imply $a^2 = 2b^2$. By the prime factorization theorem, 2 divides a^2 just twice as often as it divides a—hence an even number of times; similarly, it divides $2b^2$ an odd number of times. Therefore, $a^2 = 2b^2$ has no solution in integers.

From this "dilemma of Pythagoras" one can escape only by creating *irrational* numbers: numbers which are not quotients of integers.

Similar arguments show that both the ratio $\sqrt{3}$ of the length of a diagonal of a cube C to the length of its side, and the ratio $\sqrt[3]{2}$ of the length of a side of C to the side of a cube having half as much volume, are irrational numbers. These results are special cases of Theorem 10 of §3.7.

Further irrational numbers are π (which thus cannot be exactly $\frac{22}{7}$ or even 3.1416), e, and many others. In Chap. 14 we shall prove that the vast majority of real numbers not only are irrational, but also (unlike $\sqrt{2}$) even fail to satisfy any algebraic equation. To answer the fundamental question "*what is a real number?*" we shall need to use entirely new ideas.

One such idea is that of *continuity*—the idea that if the real axis is divided into two segments, then these segments must touch at a common frontier point. A second such idea is that the ordered field **Q** of rational numbers is *dense* in the real field, so that every real number is a *limit* of one or more sequences of rational numbers (e.g., of finite decimal approximations correct to n places). This idea can also be expressed in the statement

(2) If $x < y$, then there exists $m/n \in \mathbf{Q}$ such that $x < m/n < y$.

This property of real numbers was first recognized by the Greek mathematician Eudoxus. Thinking of $x = a:b$ and $y = c:d$ as ratios of lengths of line segments, integral multiples $n \cdot a$ of which could be formed geometrically, Eudoxus stipulated that $(a:b) = (c:d)$ if and only if, for all positive integers m and n,

(3) $na > mb$ implies $nc > md$, $\quad na < mb$ implies $nc < md$.

The two preceding ideas can be combined into a single postulate of *completeness*, which also permits one to construct the real field as a natural extension of the ordered field **Q**. This "completeness" postulate is analogous to the well-ordering postulate for the integers (§1.4): both deal with properties of infinite sets, and so are *nonalgebraic*. As we shall see, this completeness postulate is needed to establish certain essential algebraic properties of the real field (e.g., that every positive number has a square root).