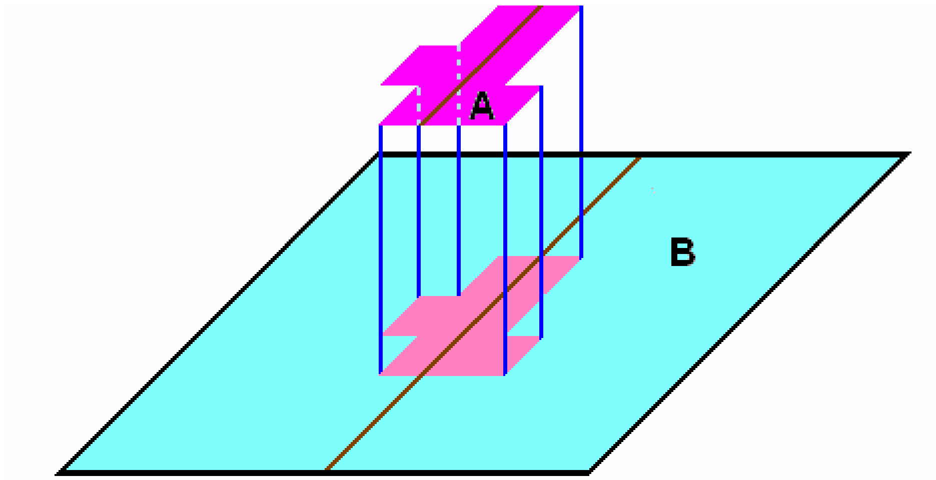
Centroids and moments

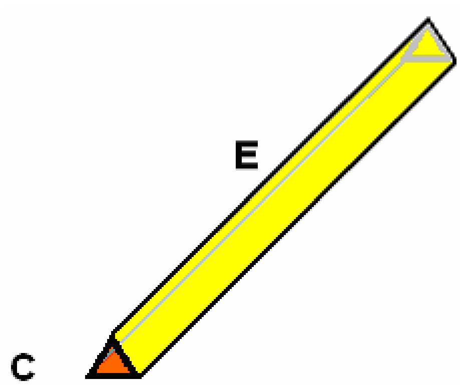
The purpose of this note is to provide some physical motivation for the standard formulas which give the center of mass of an object in the plane. Suppose **A** is a bounded planar object whose center of mass (or centroid) has coordinates (x^*, y^*), and choose positive numbers a and b such that all points on **A** belong to the solid rectangular region **B** defined by the following inequalities:

$$x^* - a \leq x \leq x^* + a$$
$$y^* - b \leq y \leq y^* + b$$

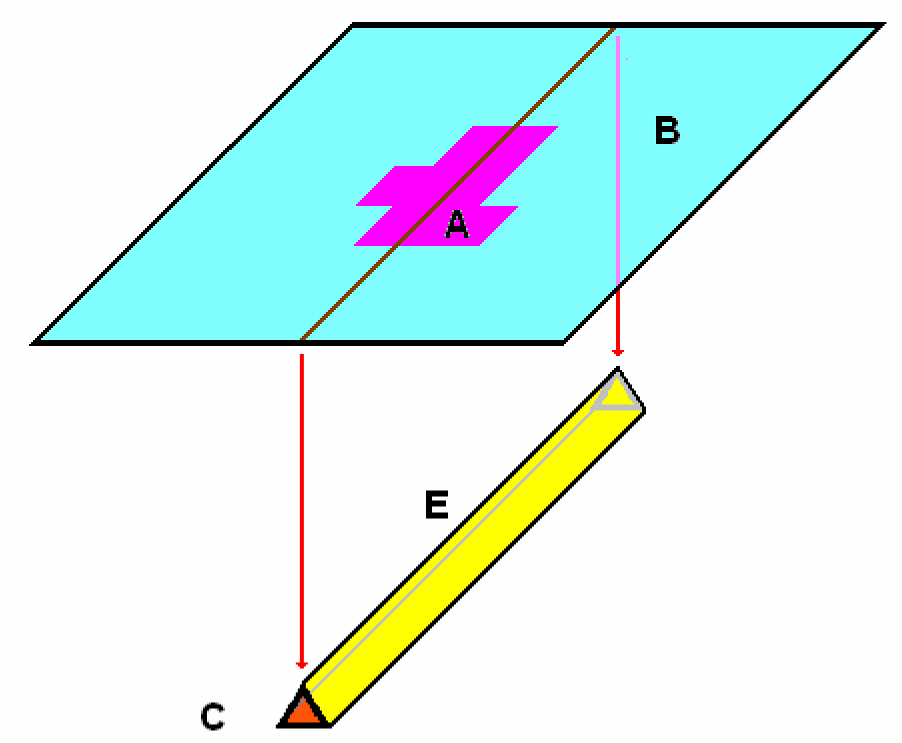
Think of **B** as a flat, firm rectangular sheet with uniform density (made of glass, metal, wood, plastic, *etc.*) such that **A** rests on top of **B**.



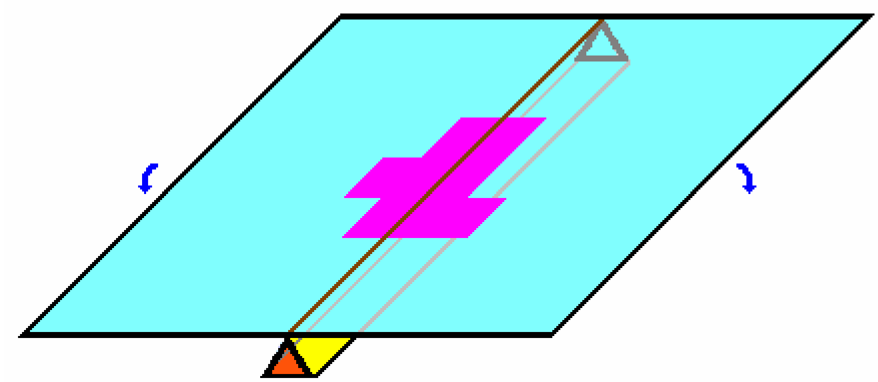
Next, suppose that we have a triangular rod **C** with equilateral ends, positioned so that one of the lateral faces is horizontal and the opposite edge **E** lies above this face.



As suggested by the figure on the next page, suppose that we now we rest **B** and **A** on the edge **E** of **C** along the vertical line defined by the equation $x = x^*$.



Since we are resting **A** and **B** along a line containing the center of mass for this combined physical system of objects**,** ***we expect that the combined object will balance perfectly*****,** not tipping either to the left or right**.**



Physically**,** this means that the total ***torque*** or ***moment*** of **A** to the right of the vertical line defined by the equation $x = x^*$ is ***equal*** to the total torque or moment of **A** to the left of the vertical line defined by the equation $x = x^*$**.**

If the mass distribution on **A** is given by the (continuous) function $\rho(x, y)$**,** then the torque equation is given by the following integral formula**:**

$$\iint_{\text{LEFT}} (x^* - x)\rho(x, y)\,dx\,dy \quad = \quad \iint_{\text{RIGHT}} (x - x^*)\rho(x, y)\,dx\,dy$$

This equation can be rewritten in the following standard form**:**

$$x^* \cdot \iint_A \rho(x,y)\,dx\,dy \quad = \quad \iint_A x\,\rho(x,y)\,dx\,dy$$

Of course, one has a corresponding equation for y^*:

$$y^* \cdot \iint_A \rho(x,y)\,dx\,dy \quad = \quad \iint_A y\,\rho(x,y)\,dx\,dy$$

On the other hand, suppose we are given a ***discrete mass distribution*** with point masses $m_i > 0$ at the points (x_i, y_i). Then the torque or moment equation for x^* is given by

$$\sum_{x(i) < x^*} (x^* - x_i) \cdot m_i \quad = \quad \sum_{x(i) < x^*} (x_i - x^*) \cdot m_i$$

(note that if $x_i = x(i) = x^*$, then the point mass at (x_i, y_i) makes no contribution to the torque on either side). As before, we can rewrite this as

$$x^* \cdot \sum_i m_i \quad = \quad \sum_i x_i \cdot m_i$$

and the latter leads directly to the standard formula for x^*. Of course, there is a similar formula for the other coordinate:

$$y^* \cdot \sum_i m_i \quad = \quad \sum_i y_i \cdot m_i$$

Application to barycentric coordinates. If we normalize our mass units so that the sum of the terms m_i is equal to 1, then the preceding equations reduce to

$$x^* \;=\; \Sigma_i\, x_i\, m_i \qquad y^* \;=\; \Sigma_i\, y_i\, m_i.$$

This observation has the following basic consequence:

FORMULA. *Suppose that we are given a discrete mass distribution, with finitely many point masses* t_i *at the points* $\mathbf{v}_i = (x_i, y_i)$, *normalized by the condition* $\Sigma_i\, t_i = 1$. *Then the center of mass* $\mathbf{v}^*$ *for this distribution is given by the barycentic coordinate expression* $\mathbf{v}^* = \Sigma_i\; t_i \mathbf{v}_i$.■