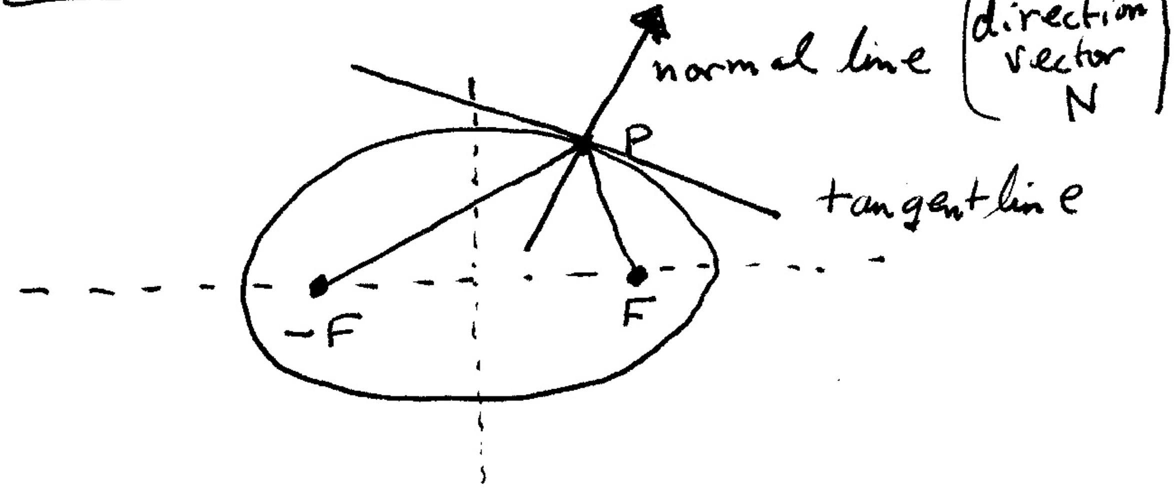
The reflection property of an ellipse



Let F and $-F$ be the focal points of the ellipse, and let P be a point on the ellipse. A basic geometrical property of the ellipse is that if a wave of sound or light is sent from one focus to P, it will bounce back and hit the other focus.

In mathematical terms, this means that the lines $(-F)P$ and FP make equal angles with the tangent line, or in vector terms the vectors $(-F-P)$ and $(F-P)$ make equal angles with the normal direction N.

It will be helpful to use the following algebraic fact:

<u>CLAIM</u> If A, B, C, D are real numbers, then $(A+B)(C-D) = (A-B)(C+D)$ if and only if $AD = BC$.

<u>PROOF</u> Expand both sides:

$$(A+B)(C-D) = AC + BC - AD - BD$$
$$(A-B)(C+D) = AC - BD - BC + AD.$$

These are equal if and only if

$$BC - AD = AD - BC$$

which happens if and only if both expressions are zero (each expression is the negative of the other). But this is equivalent to $AD = BC$.

We can now move ahead with the geometry.

Let $(x_0, y_0) = P$, and suppose that the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b > 0$. Then the focus F is $(c, 0)$ where $c^2 = a^2 - b^2$, and $-F$ is the other focus. Since the normal direction to a curve $g(x,y) = 0$ at (x_0, y_0) is $\nabla g(x_0, y_0)$, it follows that the normal direction is given by

$$N = \left(\frac{x_0}{a^2}, \frac{y_0}{b^2} \right)$$

(we have divided the gradient by 2 in order to simplify the symbolism).

In vector terms, $-F-P$ and $F-P$ make equal angles with N if and only if

$$\cos \measuredangle(-F-P, N) = \cos \measuredangle(F-P, N)$$

which translates into the equation

$$\frac{(-F-P)\cdot N}{|-F-P|\cdot|N|} = \frac{(F-P)\cdot N}{|F-P|\cdot|N|}.$$

Since $N \neq 0$ we can cancel $\frac{1}{|N|}$ so that our goal is to prove that

$$\frac{(-F-P)\cdot N}{|-F-P|} = \frac{(F-P)\cdot N}{|F-P|}$$

or equivalently

$$(\star)\quad -|F-P|\cdot\big((F+P)\cdot N\big) = |F+P|\big((F-P)\cdot N\big).$$

The next step is to insert the known values for F, P and N. Here are two especially useful simplifications:

(1) $$P\cdot N = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

(2) $$|P|^2 = x_0^2 + y_0^2 = x_0^2 + \left(b^2 - \frac{b^2 x_0^2}{a^2}\right) = x_0^2 + (a^2-c^2)\left(1 - \frac{x_0^2}{a^2}\right) = (a^2-c^2) + \frac{c^2}{a^2}x_0^2.$$

<u>USEFUL REDUCTION</u>: Since the ellipse is symmetric with respect to the two coordinate axes, it sufficies to prove the reflection property when $x_0, y_0 \geq 0$. Also since the property clearly holds if $x_0 = 0$, let's assume further that $x_0 > 0$ if necessary.

It is also useful to verify the following:

$$(3) \qquad (F-P)\cdot N < 0 < (F+P)\cdot N$$
$$(\text{assuming } x_0 > 0).$$

<u>VERIFICATION.</u> Direct computation shows that

$$(F+P)\cdot N = 1 + \frac{x_0 c}{a^2}$$
$$(F-P)\cdot N = \frac{x_0 c}{a^2} - 1$$

so the first expression is positive. However we have $0 < x_0 \leq a$ and $0 < c < a$, so the right hand side of the second expression is negative.

This has the following consequence:

(4) The signs of both expressions in ($\star$) on page 4 are equal. Hence it suffices to prove that the squares of these expressions are equal.

In other words, we need to show

$$(\star^2)\ |F-P|^2\Big((F+P)\cdot N\Big)^2 = |F+P|^2\Big((F-P)\cdot N\Big)^2.$$

Now $F\cdot P = x_0 c$, and if we combine this with (2) we see that

$$|F \pm P|^2 = c^2 \pm 2x_0 c + (a^2-c^2) + \frac{c^2}{a^2}x_0^2$$
$$= a^2 + \frac{c^2}{a^2}x_0^2 \pm 2x_0 c.$$

and we already know that

$$\Big((F\pm P)\cdot N\Big)^2 = \left(1 + \frac{x_0 c^2}{a^4}\right) \pm \frac{2x_0 c}{a^2}.$$

By the algebraic result on page 2, the validity of ($\star^2$) is equivalent to showing that

$$\left(a^2 + \frac{c^2 x_0^2}{a^2}\right) \cdot \frac{2x_0 c}{a^2} = \left(1 + \frac{x_0 c^2}{a^4}\right) \cdot 2x_0 c.$$

Since both sides are equal to

$$2x_0 c + \frac{2x_0^2 c^3}{a^4}$$

it follows that ($\star^2$) is true, and therefore this also proves the reflection property for an ellipse. $\blacksquare$