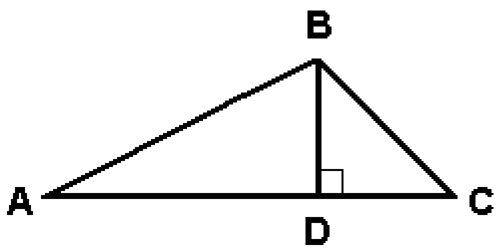
Heron's formula for the area bounded by a triangle

The standard formula for the area bounded by a triangle $\triangle$**BAC** is $\mathit{a}(\triangle\mathbf{BAC}) = \frac{1}{2}bh,$ where $d(\mathbf{A}, \mathbf{C}) = b$ and h is the length of the altitude from **B** to **AC;** hence if **D** is the foot of the perpendicular from **B to AC**, then we have $d(\mathbf{B}, \mathbf{D}) = h.$ One can also find the area bounded by $\triangle$**ABC** in terms of the lengths of its sides using a formula named after Heron (or Hero) of Alexandria (10 A.D. – 75 A.D.).

<u>Theorem (Heron's Formula)</u> *Given* $\triangle$**ABC,** *denote the lengths of its sides by* $d(\mathbf{B}, \mathbf{C}) = a$, $d(\mathbf{A}, \mathbf{C}) = b$, *and* $d(\mathbf{A}, \mathbf{B}) = c$, *and let* $s = \frac{1}{2}(a + b + c)$. *Then we have*

$$\mathit{a}(\triangle\mathbf{ABC}) = \mathrm{sqrt}\big(s(s-a)(s-b)(s-c)\big).$$

<u>Proof.</u> We know that at least two of the vertex angle measures for the triangle are less than $180,$ and without loss of generality we might as well assume that both $|\angle\mathbf{BCA}|$ and $|\angle\mathbf{CAB}|$ are less than $90;$ the other cases will follow by interchanging the roles of **A, B** and **C.** Let $\mathbf{D} \in \mathbf{AC}$ be such that **BD** is perpendicular to **AC.** Then a corollary to the Exterior Angle Theorem implies that **D** lies on the open segment **(AC).**



Let $d(\mathbf{B}, \mathbf{D}) = h,$ and let $d(\mathbf{A}, \mathbf{D}) = x,$ so that $d(\mathbf{C}, \mathbf{D}) = b - x.$ The central idea will be to solve for h in terms of $s, a, b,$ and c using the Pythagorean Theorem and some algebraic manipulations.

Applying the Pythagorean Theorem to right triangles $\triangle$**ADB** and $\triangle$**BDC,** we obtain the equations

$$x^2 + h^2 = c^2 \qquad\qquad (b - x)^2 + h^2 = a^2$$

and if we solve for h^2 we obtain the equations $c^2 - x^2 = h^2 = a^2 - b^2 + 2bx - x^2.$ Adding x^2 to each side of this equation yields $c^2 = a^2 - b^2 + 2bx,$ and if we solve this for x we find that $x = (c^2 - a^2 + b^2)/2b.$ Substituting this back into the first equation we find that

$$h^2 = c^2 - x^2 = c^2 - [(c^2 - a^2 + b^2)/2b]^2.$$

If Q denotes the area bounded by $\triangle$**ABC,** then we know that $Q = hb/2,$ and therefore we have

$$Q^2 = h^2 b^2/4 = [4c^2b^2 - (c^2 - a^2 + b^2)^2]/16 =$$

$$(2a^2c^2 + 2a^2b^2 + 2c^2b^2 - a^4 - c^4 - b^4)/16.$$

The final expression for Q^2 should look promising because it is symmetric in a, b and c. .We must now see if we can to rewrite this expression more concisely. The key to doing so is the following algebraic identity, which may be checked directly by expanding the right hand side:

$$2a^2c^2 + 2a^2b^2 + 2c^2b^2 - a^4 - c^4 - b^4 \quad =$$

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c)$$

If we let $p \; = \; a + b + c$ (the perimeter), then we have

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) \quad = \quad p(p - 2a)(p - 2b)(p - 2c)$$

and we may use these equations to write to write

$$Q^2 \quad = \quad p(p - 2a)(p - 2b)(p - 2c)/16.$$

If we now let $p \; = \; 2s$, then the preceding equation becomes

$$Q^2 \quad = \quad p(p - 2a)(p - 2b)(p - 2c)/16 \quad = \quad 2s(2s - 2a)(2s - 2b)(2s - 2c)/16 \quad =$$

$$16s(s - a)(s - b)(s - c)/16 \quad = \quad s(s - a)(s - b)(s - c)$$

and if we take square roots of both sides we obtain the area formula in the statement of the theorem. ■