6 The Extended Real Number System

6.1 The Extended Real Line

It is often convenient to make use of the *extended real line* $[-\infty, +\infty]$. This is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ obtained on adjoining to the real line $\mathbb{R}$ two extra elements $+\infty$ and $-\infty$ that represent points at 'positive infinity' and 'negative infinity' respectively. We define

$$c + (+\infty) = (+\infty) + c = +\infty$$

and

$$c + (-\infty) = (-\infty) + c = -\infty$$

for all real numbers c. We also define products of non-zero real numbers with these extra elements $\pm\infty$ so that

$$\begin{aligned} c \times (+\infty) &= (+\infty) \times c = +\infty \quad \text{when } c > 0, \\ c \times (-\infty) &= (-\infty) \times c = -\infty \quad \text{when } c > 0, \\ c \times (+\infty) &= (+\infty) \times c = -\infty \quad \text{when } c < 0, \\ c \times (-\infty) &= (-\infty) \times c = +\infty \quad \text{when } c < 0, \end{aligned}$$

We also define

$$0 \times (+\infty) = (+\infty) \times 0 = 0 \times (-\infty) = (-\infty) \times 0 = 0,$$

and

$$(+\infty) \times (+\infty) = (-\infty) \times (-\infty) = +\infty,$$
$$(+\infty) \times (-\infty) = (-\infty) \times (+\infty) = -\infty.$$

The sum of $+\infty$ and $-\infty$ is not defined. We define $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$). The difference $p - q$ of two extended real numbers is then defined by the formula $p - q = p + (-q)$, unless $p = q = +\infty$ or $p = q = -\infty$, in which cases the difference of the extended real numbers p and q is not defined.

We extend the definition of inequalities to the extended real line in the obvious fashion, so that $c < +\infty$ and $c > -\infty$ for all real numbers c, and $-\infty < +\infty$.

Given any real number c, we define

$$\begin{aligned} [c, +\infty] &= [c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p \geq c\}, \\ (c, +\infty] &= (c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p > c\}, \\ [-\infty, c] &= (-\infty, c] \cup \{-\infty\} = \{p \in [-\infty, \infty] : p \leq c\}, \\ [-\infty, c) &= (-\infty, c) \cup \{-\infty\} = \{p \in [-\infty, \infty] : p < c\}. \end{aligned}$$