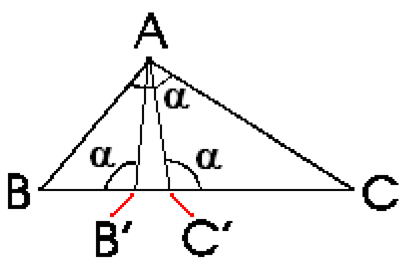
6. E. Another generalization of the Pythagorean Theorem

In **http://math.ucr.edu/~res/math153/history05f.pdf** we presented a generalization of the Pythagorean Theorem due to Pappus. As with nearly all fundamental theorems, there are many possible generalizations and analogs, and here we are concerned with one which is due to Thabit ibn Qurra. This is basically meant as a homework exercise; the solution is given on the last page. As elsewhere in the course directory, **|∠XYZ|** will denote the measure of ∠**XYZ**, and **|UV|** will denote the length of the line segment **[UV].**

Suppose that we are given △**ABC** *in which* ∠**BAC** *is obtuse* (*in other words*, **|∠BAC|** > 90°). *Let* **B′** *and* **C′** *be points on* **[BC]** *which are in the order* **B**∗ **B′**∗ **C′**∗**C** *and satisfy the condition* **|∠AB′B|** = **|∠ AC′C|** = **|∠ BAC|**. *Prove that*

$$|AB|^2 + |AC|^2 = |BC|\big(|BB'| + |CC'|\big).$$

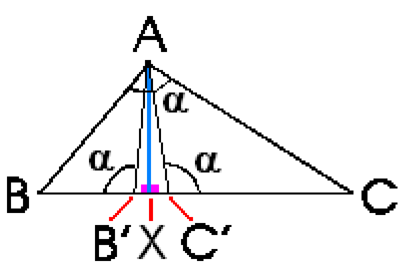
Here is a drawing to illustrate the hypotheses in the theorem:



Of course, if we have a right angle at **A** then **B′** and **C′** will be the same point, and the analog of the conclusion is merely the statement of the Pythagorean Theorem.

If we view this as an exercise, then a hint for solving it is to use similar triangles.

Technical note. Intuitively and experimentally, it is clear that we can find points with the desired properties, but we should actually **PROVE** this is possible; this proof is not needed to work the problem, but it is included for the sake of completeness. Let **X** be the foot of the perpendicular from **A** to **BC.** Since the vertex angle at **A** is obtuse, it follows from the Exterior Angle Theorem (see page 13 of **http://math.ucr.edu/~res/math133/geometrynotes3a.pdf**) that **X** must lie between **B** and **C.** If we already know that **B′** and **C′** exist then **X** will be the midpoint of **[B′C′];** we shall work backwards to find these two points.



We shall find **B′** and **C′** by figuring out what **|∠B′AX|** = **|∠C′AX|** = **½|∠C′A B′|** should be. Since **|∠BB′A|** = **|∠CC′A|** = α (where **|∠BAC|** = α), it follows that we should have $|\angle B'C'A| = |\angle C'B'A| = 180 - \alpha$, so that $|\angle C'AB'| = 2\alpha - 180$. Therefore we must have $|\angle B'AX| = |\angle C'AX| = \alpha - 90$.

We can now find the desired points **B′** and **C′** as follows: There are rays **[AU** and **[AV** on each side of the line **AX** such that $|\angle UAX| = |\angle VAX| = \alpha - 90$. In order to show that these rays meet the open segments **(XB)** and **(XC)**, we need to know that $\alpha - 90$ is less than both $|\angle BAX|$ and $|\angle CAX|$. Let $|\angle ABC| = \beta$ and $|\angle BCA| = \gamma$; then it follows that $|\angle BAX| = 90 - \beta$ and $|\angle CAX| = 90 - \gamma$, so that $\alpha - 90 = 90 - \beta - \gamma$ and thus the left hand side is less than either $|\angle BAX| = 90 - \beta$ or $|\angle CAX| = 90 - \gamma$. This means that one of the rays meets **(XB)** in some point **B′** and the other meets **(XC)** in some point **B′.** Tracing backwards through the previous discussion, we may conclude the desired formulas $|\angle BB'A| = |\angle CC'A| = \alpha$.

The preceding argument proves the existence of points **B′** and **C′** with the specified properties. The solution to the original problem, which relies on the existence of these points, appears on the next page.

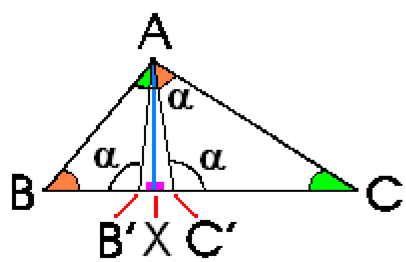
Proof of Thabit ibn Qurra's Theorem. It will be helpful to translate everything into algebra by introducing the following notation:

$$|\angle B'AB| = \delta_1 \qquad |\angle C'AX| = \delta_2$$

We then have the following equations, the first of which is true by the additivity properties of angle measure, with the others true because the angle sum of a triangle is 180:

$$\delta_1 + \delta_2 + (2\alpha - 180) = \alpha, \quad \delta_1 + \alpha + \beta = 180, \quad \delta_2 + \alpha + \gamma = 180$$

The last two of these together with $\alpha + \beta + \gamma = 180$ imply that $\delta_1 = \gamma$ and $\delta_2 = \beta$, which in turn imply that the triangles $\triangle ABC$, $\triangle B'BA$ and $\triangle C'AC$ are similar (with the indicated ordering of the vertices; see the drawing below).



These similarities imply proportionality equations; in order to state them it will be convenient to use algebraic labels for the lengths of various line segments:

$$|AB| = c, \quad |AC| = b, \quad |BB'| = p, \quad |CC'| = q, \quad |AB'| = |AC'| = r,$$
$$|XB'| = |XC'| = y, \quad |AX| = h$$

In this notation we know that $|BX| = p + y$, $|CX| = q + y$, and $|BC| = p + q + 2y$, and the similarity of the second and third triangles implies $q/r = r/p$, or equivalently $r^2 = qp$.

By the Pythagorean Theorem (applied twice), the expression $|AB|^2 + |AC|^2 = b^2 + c^2$ is equal to $2h^2 + (p+y)^2 + (q+y)^2$. Since the Pythagorean Theorem also yields the equation $y^2 + h^2 = r^2$, it follows that

$$b^2 + c^2 = 2r^2 - 2y^2 + (p+y)^2 + (q+y)^2 = p^2 + q^2 + 2py + 2qy + 2r^2.$$

Now consider the other side of the equation in the theorem's conclusion, which is equal to

$$(p + q + 2y)(p + q) = p^2 + q^2 + 2py + 2qy + 2qp.$$

Since we know that $r^2 = qp$, it follows that the two expressions are equal as asserted in the conclusion of the theorem.

FURTHER EXERCISE. Formulate and prove a similar result when all three vertex angles are acute (so that the altitude from a vertex to the opposite side meets the latter between the other two vertices by the result cited earlier).