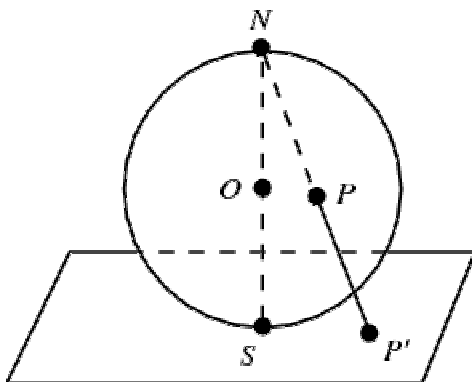


7.D. Stereographic projections

In the main notes for this unit we mentioned that the 13th century scholar Jordanus Nemorarius made a noteworthy contribution about the relationship between plane and spherical geometry, and our purpose here is to explain this more precisely with references for mathematical proofs. The levels of the proofs are slightly above the level of this course (in particular, they require input from linear algebra), but the main points can be explained relatively simply.

Geometrical and algebraic definitions

The **stereographic projection** is a function which sends points on a sphere to points on a plane, and in fact it is a $\mathbf{1} - \mathbf{1}$ correspondence between the plane and all points on the sphere except one. For the sake of definiteness take a standard sphere whose north and south poles are denoted by N and S respectively, and take the plane which is tangent to the sphere at the south pole. Then the stereographic projection of a point $P \neq N$ will be the point P' where the line joining N and P meets the tangent plane.



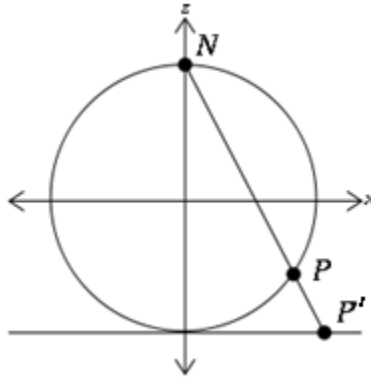
(Source: <http://mathworld.wolfram.com/StereographicProjection.html>)

Stereographic projection is frequently employed to create maps of the earth's surface centered at one of the poles (there is a picture later in this document). The construction was essentially known to Greek mathematicians like Hipparchus of Rhodes in the 2nd century B.C.E., and it probably even dates back to ancient Egyptian mathematics.

Describing the stereographic projection and its inverse in terms of coordinates turns out to be a fairly straightforward exercise in coordinate geometry. Again for the sake of definiteness, we shall take the north pole to be the standard unit vector $(\mathbf{0}, \mathbf{0}, \mathbf{1})$ and the center of the sphere to be the origin, so that the tangent plane at the south pole is has equation $z = \mathbf{1}$. Given a point (x, y, z) on the unit sphere which is not the north pole, its image is equal to

$$(\mathbf{u}, \mathbf{v}) = \left(\frac{2x}{1-z}, \frac{2y}{1-z} \right).$$

This will follow fairly directly from the picture on the next page.



(Source: <http://en.wikipedia.org/wiki/File:Stereoprojnegone.svg>)

The algebraic conditions on point P' are that $P' - N = t(P - N)$ for some positive real number t and that the last coordinate of P' is equal to -1 . These conditions uniquely determine t and yield the formula displayed on the preceding page.

A formula for the inverse function is given at the top of page 107 in the following online notes:

<http://math.ucr.edu/~res/math205A/gentopnotes2008.pdf>

Although we have focused attention on stereographic projection from the north pole, we note that one can define a stereographic projection from an arbitrary point of the sphere. Also, it turns out that the stereographic projection construction can be carried out in exactly the same manner for the n – sphere and coordinate n – space.

Properties of stereographic projections

The following basic property of stereographic projections was essentially discovered by Hipparchus:

CONFORMAL MAPPING PROPERTY. Let α and β be differentiable curves defined on the closed unit interval and taking values in coordinate n – space such that with $\alpha(0) = \beta(0)$ and the tangent vectors $\alpha'(0)$, $\beta'(0)$ are linearly independent. If the respective image curves in the n – sphere are denoted by A and B , then their tangent vectors at 0 satisfy the conditions

$$\text{angle} [\alpha'(0), \beta'(0)] = \text{angle} [A'(0), B'(0)].$$

The word “conformal” means angle – preserving, and hence the theorem states that **stereographic projection preserves the angles at which two regular smooth curves intersect.**

There is a stereographic projection map of the earth centered at the North Pole on the next page. For points that are not too far south (say above a latitude like the Tropic of Cancer), a straight line in the image plane approximates a great circle on the sphere, and one can use the conformal property and a string with two thumbtacks to give a rough estimate of the direction in which one must head in order to fly a great circle route between points on two different continents.



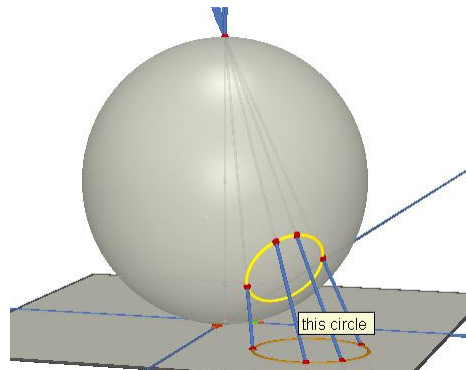
(Source: <http://www.dreamstime.com/royalty-free-stock-photos-map-of-northern-hemisphere-polar-stereographic-image7741108>)

Proofs of the Conformal Mapping Property can be found at the following online sites:

<http://people.reed.edu/~jerry/311/stereo.pdf>

<http://math.ucr.edu/~res/math153/stereo-conformal.pdf>

We now turn to another property of stereographic projections: A little experimentation indicates that **stereographic projection sends circles on the sphere to circles on the plane.**



(Source: <http://gauss.math.nthu.edu.tw/d2/qc06exe/940235/%E5%B9%BE%E4%BD%95%E4%BD%9C%E5%9C%96.htm>)

The following **YouTube** video from the American Mathematical Society illustrates this fact very effectively:

<http://www.youtube.com/watch?v=6JgGKViQzbc>

We are now in a position to explain the reason for our discussion of stereographic projections: ***Jordanus Nemorarius is given credit for the first general proof that stereographic projections have this property.*** Here is an online reference for a proof of this property:

<http://www.geom.uiuc.edu/docs/doyle/mpls/handouts/node33.html>