CONTINUOUSLY COMPOUNDED INTEREST

Exponential functions arise naturally in the theory of compound interest and some standard rules for estimating the time needed to double an investment.

Discrete compounding

Suppose that we are given a amount of money P (the *principal*) invested at a rate of $100 \times r$ per cent, compounded n times annually. After one year the investment will be worth

$$P \cdot \left(1 + \frac{r}{n}\right)^n$$

and more generally after k years the value will be

$$P \cdot \left(1 + \frac{r}{n}\right)^{kn} .$$

Although the value increases as the number of annual compoundings increases, it turns out that there is a finite upper limit to the values that cannot be exceeded no matter how many times the interest is compounded.

Continuous compounding

One way to produce an upper limit is to replace the discrete variable $1/n$ with the continuous variable x. If we let $x = 1/n$ then the expression for the value after one year becomes

$$P \cdot (1 + rx)^{1/x}$$

and the question of what happens when n gets large translates into a question about whether

$$\lim_{x \to 0} \, (1 + rx)^{1/x}$$

exists, and if so what value it takes.

Since the natural logarithm function is continuous, questions about the given limit are equivalent to questions about

$$\lim_{x \to 0} \, \frac{\log(1 + rx)}{x}$$

in the sense that the first function has a limit L_1 if and only if the second has a limit L_2, in which case we have $L_2 = \log L_1$. We can find a limit for the second function using l'Hospital's rule:

$$\lim_{x \to 0} \, \frac{\log(1 + rx)}{x} \quad = \quad \lim_{x \to 0} \, \frac{r/(1 + rx)}{1} \quad = \quad r \, .$$

This means that $L_2 = r$ which in turn implies that $L_1 = e^r$. Therefore, in the limit — where the interest is compounded every instant, or *continuously* — the value of the investment after one year is $P \, e^r$.

Similar reasoning shows that if k is a positive integer, then the value of the investment after k years is equal to $P \, e^{kr}$, and even more generally if t is a positive real number then the value after t years is $P \, e^{rt}$.

Suppose now that we want to find the time T needed for the investment to double. This means we need to solve the equation

$$2\,P \quad = \quad P\,e^{rT}$$

for T. Taking natural logarithms of both sides we obtain the equation $\log 2 = rT$, which in turn means that

$$T \quad = \quad \frac{\log 2}{r}\ .$$

Since we know that $\log 2 = 0.69314718055994530941...$ it is a straightforward exercise to find T if we know r.

The Rule of 72

The formula for finding the doubling time of an investment is precise, but often it is useful — or even necessary — to make quick estimates of the doubling time with no real opportunity to use an electronic calculator or computer. The so-called Rule of 72 is a quick and dirty way of getting an estimate of the doubling time with simple mental arithmetic. This method actually predates the introduction of logarithms and was known in the late 15^{th} century (it appears in an influential, comprehensive book by L. Pacioli on mathematical techniques).

To describe this rule, let $R = 100r$ be the annual percentage rate. Then the doubling formula can be restated as

$$T \quad = \quad \frac{69.314718...}{R}$$

and the idea behind the Rule of 72 is to round the numerator up to 72 because the latter is evenly divisible by $1, 2, 3, 4, 6, 8, 9$ and 12. This yields the approximation

$$T \quad \approx \quad \frac{72}{R}$$

which allows one to estimate T quickly by mental arithmetic if the percentage interest rate is a small positive whole number.

Virtually every approximation has a limited range in which it is reasonably accurate, so it is necessary to recognize that the Rule of 72 does not yield good approximations if the interest rate R gets too large.