Math 153
Spring 2021
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Aristotle's Theorem and coordinate geometry

Recall the following additional exercise for Unit 2:

4. Using coordinate geometry, prove the following theorem from Aristotle's *Posterior Analytics* which does not appear in Euclid's *Elements*:

> *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be distinct points in the (Euclidean) plane, and let* $0 < r < 1$*. Then the set of all points* $\mathbf{x}$ *such that* $d(\mathbf{p}, \mathbf{x}) = r \cdot d(\mathbf{q}, \mathbf{x})$ *(where "d" denotes the distance between two points) is a circle.*

Here are some suggestions for setting things up. Choose a coordinate system such that the first point is the origin and the second point has coordinates $(a, 0)$ where $a > 0$, and write out the coordinate equation corresponding to the square of the relationship in the statement of the theorem. — This exercises is meant to illustrate the fact that the *Elements* was written to cover only the basics of Greek mathematics at the time and that many known results were not included.

***Relevance to Unit* 11.** The reason for looking at this result in Unit 11 is that it illustrates how coordinate geometry provides a powerful and effective setting for solving problems ihat challenged Greek geometers and their methods.

The solution to this problem is reproduced on the following two pages.

Solution to Additional Exercise

02.4

Follow the hint, and take $p=(0,0)$, $q=(a,0)$ where $a>0$. Then the (square of the) distance equation is

$$x^2+y^2 = r^2\left((x-a)^2+y^2\right)$$

which can be rewritten as

$$(1-r^2)x^2+2r^2ax-r^2a^2+(1-r^2)y^2=0.$$

Divide this by $(1-r^2)$:

$$\left(x^2+\frac{2r^2a}{1-r^2}x\right)+y^2=\frac{r^2a^2}{1-r^2}$$

Complete the square of the x-term:

$$\left(x^2+2\frac{r^2a}{1-r^2}x+\frac{r^4a^2}{(1-r^2)^2}\right)+y^2=\frac{r^4a^2}{(1-r^2)^2}+\frac{r^2a^2}{1-r^2}$$

$$=\frac{r^4a^2+r^2a^2(1-r^2)}{(1-r^2)^2}=\frac{r^2a^2}{(1-r^2)^2}$$

Note that the right side is positive since $1-r^2>0$.

The equation above defines a circle with center $\left(\frac{-r^2a}{1-r^2},0\right)$ and radius $\sqrt{\frac{r^4a^2+r^2a^2(1-r^2)}{(1-r^2)^2}}$

$$=\frac{ra}{(1-r^2)}$$

<u>Note</u> If we let $r \to 1$ the radius goes to $+\infty$, and the circle "flattens out" into the perpendicular bisector of the closed line segment joining $(0,0)$ to $(a,0)$.