EXERCISES

1. Isolate the single real root of the following equations:

(a) $x^3 + 3x - 1 = 0$
(b) $x^3 + 3x + 2 = 0$
(c) $2x^3 + x - 1 = 0$
(d) $x^3 - 3x^2 + 4x - 1 = 0$
(e) $x^3 - 3x + 2 = 0$
(f) $x^3 + 2x^2 + 4 = 0$
(g) $x^3 + x^2 + 4 = 0$
(h) $x^3 - x^2 - x - 2 = 0$

2. Isolate the three real roots of the following equations:

(a) $x^3 - 4x - 1 = 0$
(b) $x^3 - 5x + 2 = 0$
(c) $2x^3 - 3x - 1 = 0$
(d) $2x^3 - 4x + 1 = 0$
(e) $x^3 - 3x^2 - 2x + 5 = 0$
(f) $x^3 - 7x + 7 = 0$
(g) $x^3 - 3x^2 - 2x + 2 = 0$
(h) $x^3 - 3x^2 - 13x + 7 = 0$
(i) $x^3 + 3x^2 - 3x - 3 = 0$
(j) $x^3 - 5x - 1 = 0$

3. Isolate the two real roots of the following:

(a) $x^4 - 4x + 2 = 0$
(b) $x^4 - 3x + 1 = 0$
(c) $x^4 - 7x^2 - 6x - 2 = 0$
(d) $x^4 - 8x^2 - 20x - 21 = 0$
(e) $x^4 - 7x^2 - 26x - 40 = 0$
(f) $x^4 - 14x^2 + 45x - 50 = 0$
(g) $x^4 - x^3 - 4x^2 - 3x - 1 = 0$
(h) $x^4 - 4x^3 + 5x^2 - 2x - 2 = 0$
(i) $x^4 + x^3 - 3x^2 - 8x - 6 = 0$
(j) $x^4 + x^3 - 6x^2 - 10x - 12 = 0$
(k) $x^4 + 3x^3 + 2x^2 + 2x - 4 = 0$
(l) $x^4 - 32x + 40 = 0$

4. Isolate the four real roots of the quartics:

(a) $x^4 - 15x^2 - 12x - 2 = 0$
(b) $x^4 - 22x^2 + 8x + 8 = 0$
(c) $x^4 - 40x^2 - 64x + 80 = 0$
(d) $x^4 - 11x^2 - 6x + 10 = 0$
(e) $4x^4 - 24x^2 + 8x + 3 = 0$
(f) $4x^4 - 28x^2 + 12x + 3 = 0$
(g) $x^4 + 4x^3 - x^2 - 8x - 2 = 0$
(h) $x^4 - 4x^3 - 4x^2 + 12x + 3 = 0$
(i) $x^4 - 4x^3 - 3x^2 + 8x + 2 = 0$
(j) $4x^4 - 32x^2 + 24x - 3 = 0$

7. Descartes' rule of signs (FULL COURSE). An estimate of the number P of positive roots of an equation $f(x) = 0$ may be made by counting the number V of variations of sign in the sequence of nonzero coefficients of $f(x)$. Consecutive terms of like sign may always be grouped together and the result may be written as

$$f(x) \equiv f_0(x) + f_1(x) + \cdots + f_V(x). \tag{5}$$

For example, if

$$f(x) \equiv (3x^{12} + 4x^{10}) - (2x^9 + 4x^8 + x^6) + (3x^5) - (2x^4 + 6x^2) + (11x),$$

the polynomials are $f_0 \equiv 3x^{12} + 4x^{10}$, $f_1 \equiv -2x^9 - 4x^8 - x^6$, $f_2 \equiv 3x^5$, $f_3 \equiv -2x^4 - 6x^2$, $f_4 \equiv 11x$, and $V = 4$.

The factors x of $f(x)$ and corresponding zero roots may be removed without altering the value of V, and we may assume that $f(x) \equiv a_0x^n + \cdots + a_n$ in which a_0 and a_n are *both not* zero. Then V is the number of times we change sign if we start with a_0 and pass through the coefficient sequence to a_n. Hence V is *even if a_0 and a_n have the same sign and is odd otherwise.* The number P is the number of roots between 0 and an upper bound U and $f(0) = a_n$, $f(U)$ has the same sign as a_0. By Theorem 8, P is even precisely when V is even. It follows that $V - P$ is an *even integer.*

Let us now introduce a notation $b_ix^{n_i}$ for the term of highest degree in the polynomial $f_i \equiv f_i(x)$ of formula (5), $c_ix^{m_i}$ for the term of least degree. Then

$$b_0 = a_0, \qquad n_0 = n, \qquad c_V = a_n, \qquad m_V = 0.$$

Note that in the special example above $f_2 \equiv 3x^5$ and so $b_2 = c_2$, $m_2 = n_2$.

We let r be any positive number and form the product $(x - r)f(x) \equiv (x - r)f_0 + (x - r)f_2 + \cdots + (x - r)f_V$. The term of highest degree in $(x - r)f_i$ is $b_ix^{n_i+1}$ and the term of least degree is $-rc_ix^{m_i}$. These terms have opposite signs. The leading coefficient of $(x - r)f(x)$ is the leading coefficient a_0 of $f(x)$, and so we start the count of variations of sign of $(x - r)f(x)$ with the same real number a_0. The term of least degree in every $(x - r)f_i$, for $i = 0, 1, \cdots, V - 1$, has the same sign as the term of highest degree in $(x - r)f_{i+1}$ and a sign opposite to that of the term of highest degree in $(x - r)f_i$. It follows that the number of variations in the sequence of coefficients beginning with a_0 and passing on to the term of highest degree in $(x - r)f_V$ is at least V. The constant term $-a_nr$ has sign opposite to the leading term of $(x - r)f_V$, and so the number of variations in sign for the coefficient sequence of $(x - r)f(x)$ is at least $V + 1$.

Let us now suppose that $f(x)$ is a polynomial with P positive roots $r_1, r_2, \ldots, r_P$. Then

$$f(x) \equiv (x - r_P) \cdots (x - r_2)(x - r_1)g(x).$$

The result just proved implies that, if V_0 is the number of variations for $g(x)$, then $(x - r_1)g(x)$ has at least $V_0 + 1$ variations, $(x - r_2)[(x - r_1)g(x)]$ at least $V_0 + 2$ variations, $\cdots$, $f(x)$ at least $V_0 + P$ variations. Then $V \geqq P$. We have derived the following:

DESCARTES' RULE OF SIGNS. *Let V be the number of variations of sign in the sequence of nonzero coefficients of $f(x)$ and P be the number of positive roots of $f(x)$. Then*

$$V - P$$

is a nonnegative even integer.

This rule states that there can never be more positive roots than variations of sign. It gives a precise result only when $V = 1$ and so $P = 1$, or when $V = 0$ and so $P = 0$. It may be applied to $f(-x)$ to provide an estimate of the number of negative roots of $f(x)$.

ORAL EXERCISES

Use Descartes' rule to estimate the number of positive and negative real roots of the following polynomials:

(a) $3x^5 - 2x^4 - 4x^2 - 1$
(b) $2x^7 + 4x^5 - 3x^3 - 2x$
(c) $2x^7 + 3x^4 + 5x^2 + 6$
(d) $6x^7 - 5x^4 - 1$
(e) $5x^6 + 6x^5 - 4x^3 - 3x^2 + 2x - 1$
(f) $x^7 + 3x^5 - 4x^2 - 2x + 1$
(g) $x^{10} + 9x^8 + 6x^4 - 3x^3 - 2x - 1$
(h) $x^{10} - 9x^9 + 8x^7 - 7x^6 + 8x^4 + 1$
(i) $x^4 + x^3 + x^2 - 2x + 1$
(j) $x^6 + x^5 - x^4 - x^3 + x^2 + x - 1$

8. Sturm's theorem (FULL COURSE). Let us designate any given polynomial $f(x)$ by f_0 and its derivative $f'(x)$ by f_1. Apply the division algorithm to write

$$f_0 \equiv q_1 f_1 - f_2.$$

Note that we have designated the remainder polynomial by $-f_2$. The minus sign is important.