12.E. Derivation of the Binomial Series

This is fairly standard but is included for the benefit of anyone who has not seen it previously. For an arbitrary real number a and a nonnegative integer r we write

$$\binom{a}{r} \quad = \quad \frac{a(a-1)\cdots(a-r+1)}{r!}$$

generalizing the familiar notation when n is a nonnegative integer. Note that this quantity is eventually zero if a is a nonnegative integer and r is sufficieintly large, but otherwise it is not. Furthermore, for all a and r we have the so-called "Pascal's Triangle" Identity:

$$\binom{a}{r} \quad = \quad \binom{a}{r-1} \; + \; \binom{a-1}{r-1}$$

Verifying this is a straightforward algebraic exercise.

MAIN THEOREM. *If* $|x| < 1$ *then we have*

$$\sum_{k=0}^{\infty} \binom{a}{k} x^k \quad = \quad (1+x)^a \ .$$

Sketch of derivation. We shall leave many of the details for the reader to fill in as needed. We must also assume that a is not a nonnegative integer; by the ordinary Binomial Theorem we know what happens if the exponent is a nonnegative integer, and the following discsussion breaks down because $(1+x)^n$ is a finite sum if n is a nonnegative integer.

Let $P_a(x)$ be the power series on the left hand side of the display. Then the theory of power series in first year calculus yields the following information:

(i) This series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$ by the ratio test.

(ii) Term by term differentiation yields the identity $P'_a(x) = aP_{a-1}(x)$ for all a and x such that the series converges absolutely.

(iii) Standard manipulations for convergent power series and the generalized Pascal Triangle Identity imply that $P_a(x) = (1+x)P_{a-1}(x)$.

If a is a nonzero integer, then we also know that

$$\frac{d}{dx}\,(1+x)^a \quad = \quad a(1+x)^{a-1}$$

and for more general values of a we can establish this by the logarithmic differentiation rule

$$\frac{dy}{dx} \quad = \quad y \cdot \frac{d}{dx}\,\ln y$$

provided $|x| < 1$. Furthermore, one can check directly that the identity in the theorem is valid when $x = 0$.

To verify that the identity is true in general consider the function

$$g(x) \quad = \quad (1+x)^{-a} \cdot P_a(x) \ .$$

Straigntforward application of the Leibniz rule for differentiating products implies that the derivative of this expression equals

$$(-a)(1+x)^{-a-1}P_a(x) \; + \; (1+x)^{-a} \cdot (aP_{a-1}(x))$$

and if we apply (iii) this expression becomes

$$(-a)(1+x)^{-a-1}P_a(x) \; + \; a(1+x)^{-a-1} \cdot P_a(x) \quad = \quad 0$$

so that g is a constant function. Since we have noted that $g(0) = 1/1 = 1$, it follows that $g(x) = 1$ for all x. Finally, if we multiply both sides of this equation by $(1+x)^a$ then we obtain the equation in the statement of the theorem.■