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Mathematics 153–001, Spring 2010, Examination 1

Answer Key

1. [*15 points*] In addition to the expansion $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$, one can also write

$$\frac{3}{5} \quad = \quad \frac{1}{3} \; + \; \frac{1}{a} \; + \; \frac{1}{b}$$

for positive integers $b > a$. Find a and b. [*Hint:* $4 = 3 + 1$.]

SOLUTION

We must have

$$\frac{3}{5} \; - \; \frac{1}{3} \quad = \quad \frac{1}{a} \; + \; \frac{1}{b}$$

and the left hand side is given by

$$\frac{9}{15} \; - \; \frac{5}{15} \quad = \quad \frac{4}{15} \quad = \quad \frac{3}{15} \; + \; \frac{1}{15}$$

(using the hint). But the right hand side is just

$$\frac{1}{5} \; + \; \frac{1}{15}$$

and therefore we must have $a = 5$ and $b = 15$.

2. [*20 points*] The n^{th} pentagonal number P_n is given by

$$\sum_{k=1}^{n} 3k - 2 \ .$$

Prove that $P_n = \frac{1}{2}n(3n-1)$.

SOLUTION

FIRST METHOD. We may rewrite P_n as

$$\sum_{k=1}^{n} 3k \quad - \quad \sum_{k=1}^{n} 2 \quad = \quad 3 \cdot \sum_{k=1}^{n} k \quad - \quad 2n \ .$$

Since we know that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$, it follows that the right hand side equals

$$\frac{3n(n+1)}{2} \ - \ 2n \quad = \quad \frac{n}{2}\,(3n+3-4) \quad = \quad \frac{n}{2}\,(3n-1)$$

which is the value stated in the problem.

SECOND METHOD. The formula is true for P_1 since $\frac{1}{2} \cdot 1 \cdot (3 \cdot 1 - 1) = 1$. Assume it is true for P_n; we shall use mathematical induction to prove it is true for P_{n+1}. By definition we have

$$P_{n+1} \quad = \quad P_n \ + \ 3(n+1) \ - \ 2 \quad = \quad \frac{n}{2}(3n-1) \ + \ (3n+1) \quad =$$

$$\frac{1}{2}(3n^2 - n) \ + \ \frac{1}{2}(6n+2) \quad = \quad \frac{1}{2}(3n^2 + 5n + 2) \ .$$

On the other hand, if we evaluate the expression $\frac{1}{2}k(3k-1)$ for $k = n+1$ we obtain

$$\frac{1}{2}(n+1)(3n+2) \quad = \quad \frac{1}{2}(3n^2 + 5n + 2)$$

and therefore the validity of the formula for P_k for $k = n$ implies its validity for $k = n+1$. This completes the proof by mathematical induction.

3. [*10 points*] One classical Greek construction for duplicating the cube involved the intersection points of two parabolas. If C_1 is the parabola $y = x^2$ and C_2 is the parabola $2x = y^2$, show that the meet in a point $(x, y) \neq (0, 0)$ such that one of $x,\ y$ is equal to the cube root of 2.

SOLUTION

Substitute the first equation into the second. This yields $2x = (x^2)^2 = x^4$. This equation has two solutions; namely $x = 0$ and $x = \sqrt[3]{2}$. Since $y = x^2$, the second choice yields the common point $(\sqrt[3]{2},\ \sqrt[3]{4}\,)$.

4. [*20 points*] Suppose that $\triangle ABC$ has vertex angles whose measures are 45, 60 and 75 degrees. Prove that no two sides of this triangle have equal lengths.

SOLUTION

By the Isosceles Triangle Theorem, if two sides of a triangle have equal length, then the opposite angles have equal angular measures. But no two vertex angles in $\triangle ABC$ have equal measure, so there cannot be two sides with the same measures (proof by contrapositive).

5. [*5 points*] Suppose that we are given a line L and a point x not on L. What of the statements below is implied by Euclid's Fifth Postulate? Note that that only one option can be true.

A. There are no lines through x which are parallel to L.

B. There is exactly one line through x which is parallel to L.

C. There are at least two lines through x which are parallel to L.

SOLUTION

Option "B." is implied by Euclid's Fifth Postulate and the others are not.

6. [*30 points*] For each of the statements below, indicate whether it is true of (E) Euclid or Euclid's Elements, (A) Archimedes or the results he discovered, (B) both Euclid and Archimedes, (N) neither neither Euclid nor Archimedes.

(a) Proof that there are infinitely many prime numbers.

(b) Computation of the area of a parabolic sector.

(c) Formula for the measurement of a circular arc intercepted by an angle inscribed in the given circle.

(d) Was based at the Academy in Athens.

(e) Lived after the time of Zeno.

SOLUTION

(a) "E" — This is the proof we still use today.

(b) "A" — He did this using inscribed triangles.

(c) "E" — This is mentioned explicitly in the notes.

(d) "N" — Euclid was based in Alexandria, Egypt, and Archimedes was based in Syracuse, Sicily.

(e) "B" — Zeno lived in the Fifth Century B.C.E., while Euclid was born in the Fourth Century B.C.E. and Archimedes was born in the Third Century B.C.E.