Using Descartes' Rule of Signs

Let $p(x)$ be a real polynomial with leading coefficient 1 and positive degree n.

DESCARTE'S RULE OF SIGNS

(1) If P is the number of positive roots and V is the number of changes in sign for the coefficient sequence $1, a_{n-1}, \ldots, a_0$, then $V - P$ is a nonnegative even integer.

(2) If N is the number of negative roots and V' is the number of changes in sign for the coefficient sequence $1, -a_{n-1}, \ldots, (-1)^k a_{n-k}, \ldots (-1)^n a_0$ then $V' - N$ is a nonnegative even integer. ∎

Note The second sequence is the coefficient sequence for $(-1)^n p(-x)$.

1. How many real roots does this equation have?

(1) $x^6 + x^4 - x^3 - 2x - 1 = 0$

$V = 1$, so $V - P$ is nonnegative and ~~odd~~ even, and this forces $V - P = 0$, which means $P = 1$

(2) Look at $x^6 + x^4 + x^3 + 2x - 1 = 0$

$V' = 1$, so $V - N$ is nonneg + even, so $N = 1$

Hence there are 2 real roots, 1 positive and 1 negative. ■

2. Same question for

① $x^5 + x^3 - 2x^2 + x - 2 = 0$

$V = 3$, so $P = 1$ or 3.

② $p(-x) = -x^5 - x^3 - 2x^2 - x - 2 = 0$

$V' = 0$, so $N = 0$

Hence there are $\begin{cases} 1 \text{ or } 3 & \text{positive roots} \\ 0 & \text{negative roots} \end{cases}$

③ Need to resolve the ambiguity for P.

Text book hint: Multiply by $x+1$. Then P for $p(x)$ equals P for $p(x)(x+1)$, which equals $x^6 + x^5 + x^4 - 2x^3 - x^2 - x - 2 = 0$.

For this example $V = 1 \Rightarrow p(x)(x+1)$ has exactly one positive real root $\Rightarrow$ same is true for $p(x)$.

Therefore $p(x)$ has one positive root and no negative roots. ■