Mathematics 153, Spring 2016, Examination 2

Answer Key

1. [*25 points*] Express 7/16 as a continued fraction.

Extra credit. [*10 points*] Using Descartes' Rule of Signs, show that the polynomial $x^5 - x + 1$ has at most three real roots.

SOLUTION

The continued fraction. We have

$$\frac{7}{16} = \frac{1}{\left(\frac{16}{7}\right)} = \frac{1}{2+\left(\frac{2}{7}\right)} =$$

$$\frac{1}{2+\left(1/\left(\frac{7}{2}\right)\right)} = \frac{1}{2+1/\left(3+\left(\frac{1}{2}\right)\right)} . \blacksquare$$

The extra credit portion. The original polynomial $p(x) = x^5 - x + 1$ has two changes of sign, so there are at most two positive roots. Also, the modified polynomial $p(-x) = -x^5 + x + 1$ has one change of sign, so there is at most one negative root (in fact, there is exactly one such root since the difference between the number of roots the number of sign changes and number of negative roots is nonnegative and even). Since 0 is clearly not a root because $p(0) = 1$, it follows that there are at most 1+2=3 real roots, and hence there must be at least 2 nonreal roots.$\blacksquare$

2. [*30 points*] Suppose that we are given two horizontal parallel lines in the coordinate plane defined by the equations $y = 0$ and $y = c > 0$. Let K be the locus (= set) of all points $P = (x, y)$ such that the (shortest) distance from P to the first line is **twice** the (shortest) distance from P to the second. Show that K is a pair of horizontal lines, and give the defining equations for these lines.

SOLUTION

The distance from a point $P = (x, y)$ to the first line is $|y|$ (the length of a perpendicular segment from p which meets the line), and the distance to the second line is $|y - c|$. Therefore the locus in question is defined by the equation

$$|y| \quad = \quad 2 \cdot |y - c| \ .$$

Since distances are nonnegative, this is equivalent to what we get if we square both sides: $y^2 \quad = \quad 4(y - c)^2$.

If we expand this we obtain the equation

$$y^2 \quad = \quad 4y^2 \ - \ 8yc \ + \ 4c^2$$

and if we subtract y^2 from each side we obtain the equation $0 \ = \ 3y^2 - 8yc + 4c^2$, which we can solve for y, either by finding the factoring $3y^2 - 8yc + 4c^2 = (3y - 2c)(y - 2c)$ or by using the quadratic formula. In either case we get the solution $y = 2c$ or $y = \frac{2}{3}\,c$, and it follows that the locus is the set defined by the corresponding pair of parallel lines.■

3. [*25 points*] (*i*) Find z if $z = x + yi$ is a complex number satisfying the equation

$$\frac{1}{z} \quad = \quad \frac{1}{3+4i} \; + \; \frac{1}{1-i} \; .$$

(*ii*) Given that $\log_{10} 5$ is approximately 0.7, find the number of digits in the usual base 10 expression for 5^{25}.

SOLUTION

(*i*) First compute the right hand side using the identity $w^{-1} = \overline{w}/|w|^2$, where if $w = a + bi$ is nonzero, then $\overline{w} = a - bi$ and $|w|^2 = w \cdot \overline{w} = a^2 + b^2$:

$$\frac{1}{3+4i} + \frac{1}{1-i} \quad = \quad \frac{3-4i}{25} + \frac{1+i}{2} \quad = \quad \frac{6-8i}{50} + \frac{25+25i}{50} \quad = \quad \frac{31+17i}{50}$$

The equation stipulates that the latter equals $1/z$, and therefore we have

$$z \quad = \quad 50 \cdot \frac{31-17i}{31^2+17^2} \quad = \quad 50 \cdot \frac{31-17i}{1250}$$

which we can simplify further to $(31 - 17i)/25$.■

(*ii*) We have $\log_{10} 5^{25} = 25 \log_{10} 5 \approx 25 \cdot 0.7 = 17.5$, so that $5^{25} \approx 10^{17.5}$. This means that the base 10 expansion of 5^{25} has 18 digits.■

4. [*25 points*] Given an odd number $2k+1$ where $k \geq 1$, find a positive integer n such that $n^2 + (2k+1)^2 = (n+1)^2$; *i.e.*, $(2k+1,\ n,\ n+1)$ is a Pythagorean triple.

SOLUTION

If we expand the equation for the Pythagorean triple we obtain

$$n^2 \ + \ 4k^2 \ + \ 4k \ + \ 1 \ \ = \ \ n^2 \ + \ 2n \ + \ 1$$

which reduces to $4k^2 + 4k = 2n$ if we cancel like terms. If we divide both sides by 2, we obtain the equation $n = 2k^2 + 2k$.■

This immediately yields many standard triples, including $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$ and $(9, 40, 41)$.

5. [*15 points*] For each of the following, state whether Newton, Leibniz, or others are more closely identified with the topic:

(i) Base 2 (binary) expansions of numbers.

(ii) Formulas for integrating polynomial functions.

(iii) Infinite series expansion of $\sqrt{1+x}$ for $|x| < 1$.

SOLUTION

(i) Leibniz

(ii) Others (*e.g.*, Saint-Vincent, Wallis)

(iii) Newton

6. [*30 points*] Put three of the following lists of mathematicians in chronological order. Extra credit of 10 points is possible if all four are placed in correct order. It will suffice to give an answer of the form $PQRS$, where $\{P, Q, R, S\} = \{1, 2, 3, 4\}$ and the number associated to a name is the name's position in the alphabetical ordering.

(a) Eratosthenes, Diophantus, Heron, Pappus

(b) Al-Khwarizmi, Aryabhata, Bhaskara, Khayyam

(c) Cardano, Fermat, Regiomontanus, Viète

(d) Bernoullis (John and Jacob), Cauchy, Euler, Weierstrass

(e) Berkeley, Napier, Recorde, Wallis

SOLUTION

(a) Eratosthenes—Heron—Diophantus—Pappus

(b) Aryabhata—Al-Khwarizmi—Khayyam—Bhaskara

(c) Regiomontanus—Cardano—Viète—Fermat

(d) Bernoullis (John and Jacob)—Euler—Cauchy—Weierstrass

(e) Recorde—Napier—Wallis—Berkeley