Exercises on Fibonacci numbers

Burton, p. 293 #2

(a) Show
$$F_1 + F_3 + \dots + F_{2n-1} = F_{2n}.$$

Follow the hint: $F_1 = F_2$
$F_3 = F_4 - F_2$ etc.

Then $F_1 + F_3 + \dots + F_{2n-1} =$

$$F_2 + (F_4 - F_2) + (F_6 - F_4) + \dots + (F_{2n} - F_{2n-1}) = F_{2n}$$

(b) $F_2 + \dots + F_{2n} \overset{??}{=} F_{2n+1} - 1$ (To be proved)

Follow the hint and use the identity
$$F_1 + F_2 + \dots + F_{2n} = F_{2n+2} - 1$$
(proved on page 290 of Burton)

Subtract the identity in (a) from this, so that
$$F_2 + F_4 + \dots + F_{2n} = \left(F_{2n+2} - 1\right) - F_{2n}$$

Since $F_{2n+2} = F_{2n} + F_{2n+1}$, we get the desired identity.

(c) Show that

$$F_1 - F_2 + F_3 - F_4 \cdots + (-1)^{n+1} F_n =$$

$$(-1)^{n+1} F_{n-1} + 1.$$

If $n = 2$ the formula is true because the left and right sides are both zero. Proceed by induction, assuming the result for $k \geq 2$. Let A_k denote the alternating sum on the left hand side. Then

$$A_{k+1} = A_k + (-1)^{k+2} F_{k+1} = \left(\begin{array}{l}\text{use the induction} \\ \text{hypothesis}\end{array}\right)$$

$$(-1)^{k+1}\left[F_{k-1} - F_{k+1}\right] + 1 = \left(\begin{array}{l}\text{use} \\ F_{m+2} = F_{m+1} + F_m\end{array}\right)$$

$$(-1)^{k+1}\left[F_{k-1} - (F_k + \underset{\uparrow \text{MINUS}}{F_{k-1}})\right] + 1 =$$

$$(-1)^{k+2} F_k + 1$$ which is what we want to complete the inductive step.

Burton, p. 293 #7(c)

Show that $4 \mid F_n \iff 6 \mid n$.

The general result implies that $6 \mid n$ $\iff 8 = F_6 \mid F_n$. We need to show that if 4 divides F_n then so does 8. The key to doing this is the following:

CLAIM: If we set $F_0 = 0$, then for all $n \geq 0$ the difference $F_{n+6} - F_n$ is divisible by 4.

Let's assume this is true. Then it follows that F_n and F_{n+6k} have the same remainder when divided by 4, where k is any positive integer. So the remainders are just those arising from $F_0, \ldots, F_5$. These are $0, 1, 1, 2, 3, 1$. In particular, F_n is even only if $3 \mid n$ (which we knew), and for all k the number F_{6k+3} has a remainder of 2 upon division by 4, so 4 does not divide F_{6k+3} for any k. Thus $4 \mid F_n \Rightarrow 6 \mid n$.

Verification of the claim. We do this by induction on n. If $n=0$ then $F_0=0$ and $F_6=8$, so the statement is true in the bottom case. ~~Suppose that~~ If $n=1$, then $F_1=1$ and $F_7=13$, so the statement is again true.

Suppose we know the statement for $n=k\geq 2$. Then $F_{k+6}-F_k=4a$ and $F_{k+5}-F_{k-1}=4b$ for some integers a, b. Hence

$$F_{k+7}-F_{k+1}=(F_{k+6}+F_{k+5})-(F_k+F_{k-1})=$$

$$(F_{k+6}-F_k)+(F_{k+5}-F_{k-1})=4a+4b=4(a+b),$$

proving the inductive step.

Not in Burton

Suppose we take a Fibonacci-like sequence with $u_n = u_{n-1} + u_{n-2}$ $(n \geq 2)$ such that $u_0 = a$ and $u_1 = b$. Prove that 11 divides $u_0 + \dots + u_9$. (a & b are integers).

Solution

Start writing down terms

$$\left.\begin{aligned} u_0 &= a \\ u_1 &= b \\ u_2 &= a + b \\ u_3 &= a + 2b \\ u_4 &= 2a + 3b \\ u_5 &= 3a + 5b \end{aligned}\right\}$$

See the pattern? The coefficients are Fibonacci numbers! In fact,

$$u_n = F_{n-1}\, a + F_n\, b$$

Suppose we add up $u_0, \dots u_n$.

$$\sum_{j=0}^{n} u_j = \left(1 + \sum_{k=1}^{n-1} F_k\right) a + \left(\sum_{k=1}^{n} F_k\right) b.$$

We want to take $n = 9$ (ten terms)

By the formula on page 290 of Burton

$F_1 + \dots + F_8 = F_{10} - 1 = 55 - 1 = 54.$ *

$F_1 + \dots + F_9 = (F_{10} - 1) + F_9 = F_{11} - 1 = 88$ *

Therefore

$$w_0 + \dots + w_{10} = (54+1)a + 88b = 11(5a + 8b)$$

so that the sum is divisible by 11.

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* Reference Table of Fibonacci Numbers

$F_1 = 1$	$F_6 = 8$	$F_{11} = 89$
$F_2 = 1$	$F_7 = 13$	$F_{12} = 144$
$F_3 = 2$	$F_8 = 21$	$F_{13} = 233$
$F_4 = 3$	$F_9 = 34$	$F_{14} = 377$
$F_5 = 5$	$F_{10} = 55$	$F_{15} = 610$

<u>COROLLARY</u> For all n we have that

$$\sum_{k=0}^{9} F_{n+k} \text{ is divisible by } 11.$$