6. Mathematics of Asian and Arabic civilizations — I

(Burton, **5.3, 5.5, 6.1**)

Every civilization needs to develop some mathematical knowledge in order to succeed, and several other ancient civilizations went quite far in producing substantial amounts of mathematics. Not surprisingly, many fundamental mathematical ideas were discovered independently in each of these civilizations, but there were also noteworthy differences in the organization and emphasis of the subject, and in many cases one civilization discovered things which others did not. We shall begin this unit with brief discussions of mathematics in the civilizations of China and India. As indicated at the beginning of this course, we shall not try to do this comprehensively, but instead we shall try to focus on relatively unique features of mathematics in these civilizations and on advances which influenced the development of the mathematics we have in our contemporary civilization. We shall pay particular attention to the contributions of Indian civilization in these notes because the work of Indian mathematicians has had a particularly strong impact on mathematics as we know it today.

Mathematical activity in China

It seems clear that significant Chinese work in mathematics goes back at least 3000 years and probably at least a millennium longer, but our knowledge prior to about 100 B.C.E. is sketchy and often quite speculative. However, by that time the Chinese mathematics was already quite well developed. The level of Chinese mathematical knowledge and ability at the time can be seen from the contents of ***The Nine Chapters on the Mathematical Art***, which was apparently written in the 1st century B.C.E. and was extremely influential (with major commentaries by Liu Hui, 220 – 280, and others). The ***MacTutor*** site contains a brief but very informative survey of this work's contents:

http://www-history.mcs.st-and.ac.uk/HistTopics/Nine_chapters.html

One important difference between Chinese and Greek mathematics involved the role of logic. Chinese mathematics did not use deductive logic as the framework for the subject, and the interest was more directed towards solving wide ranges of problems and obtaining numerically accurate solutions than to studying the subject for its own sake or describing solutions qualitatively. The Chinese algorithm for extracting square roots, which was widely taught in American schools during the first few decades of the 20th century, illustrates these features of Chinese mathematics. Here are online references for the algorithm and its justification:

http://www.homeschoolmath.net/teaching/square-root-algorithm.php

http://www.homeschoolmath.net/teaching/sqr-algorithm-why-works.php

However, there was also interest in some topics which were not intrinsically practical. For example, the Chinese were fond of patterns, and at some very early point they discovered the existence of magic squares, which are square matrices of positive integers such that the sums of the rows, columns and diagonals are all equal to some fixed value. Further discussions of this topic are in the following two online files:

http://math.ucr.edu/~res/math153/oldmagicsquare.pdf

http://math.ucr.edu/~res/math153/oldmagicsquare2.pdf

Early Chinese mathematics also included **(1)** relatively advanced methods for approximate numerical solutions (versions of ***Horner's method***, named after W. G. Horner, 1786 – 1837; see http://mathworld.wolfram.com/HornersMethod.html), **(2)** systematic procedures for solving systems of linear equations, and **(3)** the following basic class of problems involving a result known as the ***Chinese Remainder Theorem***:

Let p and q be two relatively prime positive integers, and suppose that long division of a third positive integer n by p and q yields remainders of a and b respectively. What value(s) must n take?

For example, if $p = 3$ and $q = 5$, and the remainders are 2 and 3 respectively, then n must have the form $8 + 15m$ for some integer m. The first known examples of such problems appeared in a highly influential mathematical manual written by Sun Zi (also called Sun Tzu, 400 – 460; distinct from the military strategist Sun Tzu, 544 B.C.E. – 496 B.C.E.). The file http://math.ucr.edu/~res/math153/history06c.pdf discusses problems of this sort and presents methods for solving them.

Before moving ahead, we shall give another reference for Horner's method; namely, pages 174 – 177 from the following older college algebra textbook:

> **A. A. Albert.** ***College Algebra*** (Reprint of the 1946 Edition).
> University of Chicago Press, Chicago, IL, 1963.

The peak period of Chinese mathematics took place during the 13th century and early 14th century. One of the most prominent figures of the time was Qin Jushao (1202 – 1261), whose ***Mathematical Treatise in Nine Sections*** covers many of the topics discussed before at more sophisticated levels, including polynomial equations with degrees up to 10 and some equations of Diophantine type which went further than the problems which Diophantus is known to have considered. Solutions to complicated systems of equations also figured in the research of contemporaries including Li Zhi (also called Li Yeh, 1192 – 1279), Yang Hui (*c.* 1238 – *c.*1298) and Zhu Shijie (1260 – 1320). Yang Hui also wrote extensively on magic squares and mathematical education.

Chinese mathematical activity declined after the peak period, but it never really ended; elements of Chinese mathematical tradition continued after the infusion of Western mathematical knowledge beginning with the missionary work of M. Ricci (1552 – 1610).

Finally, we note that the online site http://aleph0.clarku.edu/~djoyce/mathhist/china.html contains a somewhat more detailed overview of Chinese mathematical history.

Mathematical activity in India

Indian mathematics has a long and interesting history, probably going back at least 4000 years with a sequence of three distinct eras during the ancient period (the **Harappan** or **Indus Valley era** until about 1500 B.C.E., the **Vedic era** from about 1500 B.C.E. until about 500 B.C.E., and the **Jaina era** from about 500 B.C.E. until about 500 A.D.). Although there may have been some mathematical interactions between Indian mathematics and Greek or Chinese mathematics, the Indian approach to the subject also contained concepts and ideas that were not well developed by either of the other civilizations. In keeping with the focus of this course, we shall begin our discussion of Indian mathematics with comments on its distinctive features and its advances which ultimately had a major impact on modern mathematics.

Logical structure played a more significant role in Indian mathematics than in Chinese mathematics, but it was definitely not comparable to the place of logic in Greek mathematics. Another noteworthy difference between Indian and Greek mathematics is that Indian mathematicians were less troubled by distinctions between rational and irrational numbers, and in fact they were far more willing to consider still other concepts including negative numbers, zero, and even infinite objects in some cases. Ancient Indian mathematics is also distinguished by its extensive use of poetic language in its mathematical writings; one apparent reason for this was relative ease of memorization. In a similar vein, the mathematical problems in Indian mathematical writings often were placed into highly imaginative settings.

Although studies of grammatical structure have only recently become linked to the mathematical sciences, linguists like N. Chomsky (1928 – , now more widely known for his controversial political views), computer scientists like J. Backus (the developer of **FORTRAN,** 1924 – 2007) and many others have forcefully demonstrated the connection between the two subjects and their importance for each other. In view of this, the extensive work of Pāṇini (*c.* 520 – *c.* 460 B.C.E.) on Sanskrit grammar definitely deserves to be included as a contribution to mathematics as we know it today; his studies strongly anticipated much of the 20th century work on the grammatical structures that is fundamentally important for operating computers (and one important concept is known as the ***Panini – Backus normal form***). Somewhat later writings of Pingala (probably between 400 and 100 B.C.E.) on prosody (PROSS – o – dee = the rhythm, stress, and intonation of language, usually in its spoken form) contains the first known description of a binary numeral system. The study of language patterns led to substantial work on combinatorial (counting) problems, and in several respects Pingala's results appear to have anticipated important later developments.

Probably the best known, and most widely used, legacy of Indian mathematics is the base **10** place value numeration system in use today, with nine basic digits arranged in sequences and the roles of the digits determined by their placement. With the ultimate incorporation of zero into the framework, the nine digit system grew to the ten digit notation that has become standard worldwide. The impact of this discovery is stated clearly and concisely in the following quotation from P. – S. Laplace (1749 – 1847):

> The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Apollonius.

Some comparisons with numeration in other civilizations may be enlightening. The Babylonians actually had a base **60** place value numeration system, but the Greek numeration system did not; in fact, it was more similar in structure to Roman numerals, where a number like **234** was written using a special symbol combination for **60** (**CC**), another special symbol combination for **30** (**XXX**), and yet another special symbol combination for **4** (**IV**). However, the Greek conventions only involved single symbols for the hunrdeds, tens and units terms, using their alphabet at the time for the symbols. A chart giving the symbols for the various numbers is on the next page. Three symbols correspond to letters which are no longer used in the language; namely, Ϝ (***vau*** or ***digamma***) or ς (***stigma***) for **6,** ϟ or ϙ (***qoppa***) for **90,** and ϡ (***sampi***) for **900.**

Letter	Value	Letter	Value	Letter	Value
α	1	ι	10	ρ	100
β	2	κ	20	σ	200
γ	3	λ	30	τ	300
δ	4	μ	40	υ	400
ε	5	ν	50	φ	500
Ϝ or ς	6	ξ	60	χ	600
ζ	7	ο	70	ψ	700
η	8	π	80	ω	800
θ	9	ϟ or ϙ	90	ϡ	900

(***Source:*** http://en.wikipedia.org/wiki/Greek_numerals)

For example, the traditional representation of the celebrated number **666** in Greek versions of the New Testament was χξς′; four digit numbers were formed by using the symbols for units in the thousands place with a special accent sign — to illustrate this, we note that the number **2012** would have been written as ͵βιβ′.

Historians have varying opinions on exactly when the Indian place value system was developed ranging from the 1st to 5th century A.D., with some evidence pointing to the 2nd or 3rd century A.D., but ancient Indian mathematics has very little to say about exactly when discoveries were made or who made them. There is a manuscript (*Lokavibhâga*, or *Parts of the Universe*) that is known to have been written no later than 458 A.D. and explicitly discusses a place value system including zero, and an analysis of the text indicates the underlying ideas had been known for some time (see Chapter 24, and especially pages 420 – 421, of the book by Ifrah cited in Unit 1). However, it took several centuries before this system was widely used (this is also discussed in the book by Ifrah).

The idea of using nine or ten digits also appears explicitly as a well – known technique in the writings of Āryabhaṭa the Elder (476 – 550; the first Indian satellite, launched in 1975, was named after him) and he is viewed as the earliest mathematical contributor to the **classical era** of Indian mathematics from about 500 to 1200 A.D., so we shall begin with his work. The surviving mathematical work of Aryabhata is contained in a manuscript called the ***Āryabhaṭīya***, which is written entirely in verse and also covers other subjects besides mathematics. As noted before, there is a reference describing a numbering system like the one we use today, and the mathematical portion of the work also contains results on integral solutions to Diophantine equations of the first and second degree. Trigonometry also played a very significant role in Indian mathematics, and in fact modern trigonometry follows the Indian approach — which is based upon the **sine** function — rather than the Greek approach based upon the chord function. The trigonometric tables in the ***Aryabhatiya*** compute trigonometric functions for angles with a basic increment of 3.75 degrees (= $^1/_{24}$ of a right angle).

One of the most important figures in Indian mathematics was Brahmagupta (598 – 670), whose writings (most notably the ***Brāhmasphuṭasiddhānta***) contain many important new and far – reaching ideas. We shall list a few of them:

1. He explicitly recognized that Diophantine equations can have many solutions.
2. He used nine or ten symbols to write numbers (Aryabhata used an older alphabetic system).
3. He had no reservations about working with negative numbers and irrationals.
4. His work recognizes the concept of zero, although the first known explicit use of a symbol for zero in written Indian mathematics does not occur until late in the 9th century.
5. He devoted a great deal of effort to analyzing Diophantine equations like the previously discussed Pell equation $x^2 = 1 + ay^2$. Further results on this equation due to Bhāskara (1114 – 1185) are mentioned below; Brahmagupta's main contribution was to give a method for constructing new solutions out of previously known ones (a method for doing so is given in **http://math.ucr.edu/~res/math153/history06b.pdf**).

Brahmagupta's writings also treat geometrical topics, but some of his conclusions are extremely inaccurate and far below the quality of his algebraic results. However, one particularly noteworthy geometric result due to him is an area formula for a quadrilateral that can be inscribed in a circle (see Exercise 6 on page 193 of Burton). Further information on proofs for this result may be found at the following online sites:

http://jwilson.coe.uga.edu/emt725/brahmagupta/brahmagupta.html

http://en.wikipedia.org/wiki/Brahmagupta's_formula

The work of Mahāvīra (or Mahaviracharya, *c.* 800 – *c.* 870) clarified and extended the ideas of Aryabhata and Brahmagupta, and his only surviving work (***Ganit Saar Sangraha***) is the earliest known Indian text devoted entirely to mathematics. Like other mathematicians from the classical era, he discussed arithmetic operations involving zero, and like the others he found the concept of division by zero to be troublesome. Brahmagupta had tried to explain division by zero, but he did not get very far and mistakenly asserted that the indeterminate form $0/0$ should equal 0. Mahavira mistakenly suggested that division by zero had no effect on the number being divided. More than two centuries would pass until Bhāskara, the next epic figure in the history of Indian mathematics, suggested that "if one divides a finite nonzero number by zero, the result is infinity," which is now viewed as one of the best possible intuitive descriptions (however, this statement can lead to paradoxes and fallacious proofs if it is not used carefully). For the sake of completeness, here are some mathematically accurate online references concerning division by zero:

http://mathworld.wolfram.com/DivisionbyZero.html

http://mathworld.wolfram.com/AffinelyExtendedRealNumbers.html

http://mathworld.wolfram.com/ProjectivelyExtendedRealNumbers.html

http://en.wikipedia.org/wiki/Extended_real_number_line

http://en.wikipedia.org/wiki/Projectively_extended_real_numbers

We turn now to Bhāskara (also known as Bhaskara II or Bhaskarachariya), who is often viewed as the most important figure in classical Indian mathematics. His main work, ***Siddhānta Shiromani***, consists four parts (*Lilāvati*, *Bijaganita*, *Grahaganita* and *Golādhyāya*) which deal with arithmetic, algebra, astronomy and spheres.

The concept of zero is far more explicit in Bhaskara's work than in earlier writings, and as noted above he made a crucial advance in our efforts to understand the issues involving division by zero. Also, he clearly understood that quadratic equations have two roots; one verbal problem in his writings yields the equation $x^2 = 64(x-12)$, and he notes that 16 and 48 are both valid solutions. His mathematical writings also used the decimal system methodically to an unprecedented degree.

Some of Bhaskara's deepest discoveries involve Pell's equation $px^2 + 1 = y^2$. His numerical results on this equation include the following: For $p = 61$ there is a solution $x = 226153980$, $y = 1776319049$, and for $p = 67$ there is a solution $x = 5967$, $y = 48842$. More generally, Bhaskara developed a very elegant algorithmic "cyclic method" for finding a minimal solution to Pell's equation when p has no square divisors except 1; descriptions of this method appear on pages 223 – 225 of the textbook by Katz and pages 10 – 11 of the following online document:

http://www.math.ucla.edu/~vsv/gamelin.pdf

Although Bhaskara's method is relatively simple and efficient, a proof that it always works was not given until 1929 in the following paper:

> **K. A. A. Ayyangar**, ***New light on Bhaskara's Chakravala or cyclic method of solving indeterminate equations of the second degree in two variables.*** Journal of the Indian Mathematics Society **18** (1929), 232 – 245.

The following paper contains still further information:

> **C. – O. Selenius.** ***Rationale of the chakravāla process of Jayadeva and Bhāskara II.*** Historia Mathematica **2** (1975), 167 – 184.

Bhaskara also made noteworthy contributions in other areas. For example, in his work on astronomy he broke new ground in studying trigonometry for its own sake rather than for its computational value, and he had many insights which were rudimentary versions of key facts in differential and integral calculus.

The classical period in Indian mathematics basically ended with Bhaskara; invasions by Islamic armies from Afghanistan during the late 12th and early 13th centuries dramatically changed the course of Indian history, especially outside the south of the subcontinent (for example, see http://www.infoplease.com/ce6/history/A0815061.html). However, some important mathematical activity continued in the south, most notably in Kerala at the tip of the peninsula (see http://math.ucr.edu/~res/math153/map-India.pdf). The Kerala School of mathematics built upon the work of Bhaskara on astronomy and trigonometry, and the most renowned achievements were strong results on infinite series related to trigonometric functions. For example, Madhava of Sangamagrama (1340 – 1425) discovered the standard infinite series for $\arctan x$, and subsequently Nilakantha Somayaji (1444 – 1544) found an infinite series for $\pi/4 = \arctan 1$ that converges much more rapidly than the standard series for $\arctan 1$ (more details and some references are given in the next unit). In many respects the results of the Kerala school foreshadowed the development of calculus (although many key ingredients in calculus were missing and claims that their discoveries were transmitted to the Western world around 1600 are not supported by direct evidence; however, it is conceivable that Western missionaries or merchants might have passed along information about the findings of the Kerala school). Aside from the Kerala School's results on infinite series and trigonometric functions, an early version of the Mean Value Theorem in differential calculus was obtained by Parameshvara (1370–1460).

The original discoveries of the Kerala school appear to have dried up in the early years of the 17th century, but the school itself survived for about another two centuries, with the final activity just a few decades before the British established their rule over the entire subcontinent during the middle of the 19th century. Shortly afterwards, Western mathematics began to exert a strong influence on the Indian subcontinent. However, mathematics in India retained some traditional features at least for a while, and some historians have cited the extraordinary mathematical studies of S. A. Rāmānujan (1887 – 1920) as an example of this phenomenon. More information about this unique mathematician and his work can be found in the following online references:

http://en.wikipedia.org/wiki/Srinivasa_Ramanujan

http://www.usna.edu/Users/math/meh/ramanujan.html

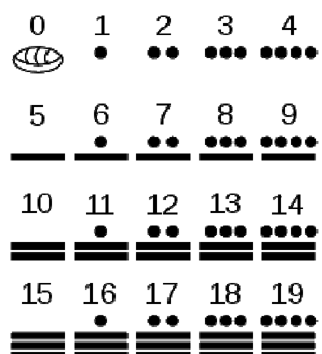
http://scienceworld.wolfram.com/biography/Ramanujan.html

Since the most far – reaching consequences for modern mathematics were transmitted to the Western World indirectly through Arabic/Islamic civilization, we shall move on to the latter after a few final remarks in this half of the unit.

Remarks on other non – Western cultures

We shall limit this discussion to a few unique and particularly striking achievements.

Given that it took many civilizations a long time to recognize the concept of zero and to use it in their numbering systems, it is remarkable that zero was included in Mayan and pre – Mayan numeration systems more than 2000 years ago. The Mayans had a very complicated and well – developed calendar which used the zero concept, and some objects have Mayan calendar dates corresponding to approximately 30 B.C.E.. A chart depicting their base 20 number system is given below.



(Source: http://upload.wikimedia.org/wikipedia/commons/thumb/1/1b/Maya.svg/220px-Maya.svg.png)

Some features of **Japanese mathematics** also deserve to be mentioned. Near the beginning of the 17th century, a sophisticated and highly original school of mathematics developed while the country was largely isolated from the outside world (1633 – 1868). This work was largely based upon Chinese mathematics, but it went far beyond the latter in some respects and had its own style. One of the most celebrated mathematicians of this school was Takakazu (or Kōwa) Seki (1642 – 1708). He did a considerable amount of work on infinitesimal calculus and Diophantine equations, and he also made important discoveries related to matrix algebra (more precisely, the theory of determinants). Seki was a contemporary of Gottfried Leibniz and Isaac Newton, but his work was entirely independent. Important aspects of Seki's work were disseminated and carried further by Takebe Katahiro (1664 – 1739), also known as Takebe Kenkō; other prominent names in this school include Naonobu Ajima (1732 – 1798), also known as Ajima Chokuyen, and Wada Yenzō Nei (1787 – 1840). With the opening of Japan to the outside world in the middle of the 19th century, the influence of Western mathematics led to the rapid decline and assimilation of the Japanese school. The following book is (still) an excellent source of information about this aspect of mathematical history:

D. E. Smith and Y. Mikami. *A History of Japanese Mathematics* (Reprint of the 1914 edition). Dover Publications, New York, 2004.

Some additional references

Here are two further references on Asian mathematics from non – Western viewpoints, one of which is a history of Chinese mathematics and another of which studies possible Indian influences in the development of calculus by European mathematicians.

Yǎn Lǐ and Shíràn Dů. *Chinese mathematics: a concise history* (Translated by J. N. Crossley and A. W. – C. Lun, with a foreword by J. Needham). Oxford University Press, New York NY, 1987.

There is an informative review of this book and another on the same topic (Martzloff, ***History of Chinese Mathematics***, with an English translation by S. S. Wilson that was published by Springer – Verlag in 1997). It was written by K. Chemla and appeared in *The British Journal for the History of Science* **23** (1990), pp. 493 – 495.

George Gheverghese Joseph. *The Crest of the Peacock: Non – European Roots of Mathematics* (Third Edition). Princeton University Press, Princeton, NJ, 2011.

This is a very interesting, engaging and controversial book whose strongly stated thesis is that the mathematical activities of non – Western cultures have been unfairly dismissed or discounted to a great extent. In particular, Chapter 10 contains a detailed discussion of the possibility that the work of mathematicians from the Kerala School in India was known to European mathematicians who developed calculus and its forerunners in the 18th century (the circumstantial evidence has been mentioned, and the book develops this argument further). Some thoughtful criticisms of the book appear in a few of the following **amazon.com** reviews:

http://www.amazon.com/product-reviews/0691006598/ref=cm_cr_dp_syn_footer/277-5801994-5953420?k=The%20Crest%20of%20the%20Peacock%3A%20The%20Non-European%20Roots%20of%20Mathematics&showViewpoints=1

Addenda to this unit

There are five separate items**.** The first document (**6.A**) discusses the contributions of Arabic mathematics and some highly critical comments on this topic that have been published recently**,** the second (**6.B**) describes a method for propagating one solution of Pell's equation into an infinite family of solutions**,** the third (**6.C**) presents general techniques for solving examples associated to the Chinese Remainder Theorem**,** the fourth (**6.D**) contains remarks on the sometimes confusing fact that the product of two negative numbers is positive**,** and the last (**6.E**) discusses a generalization of the Pythagorean Theorem which was formulated by the Arabic mathematician Thabit ibn Qurra**.**