7.C. The last problem in Fibonacci's *Book of Squares*

The surviving copy of *Liber quadratorum* ends very abruptly while discussing a generalization of the final problem, which was proposed by Master Theodore, who was a scholar in the court of (Holy Roman) Emperor Frederick II. Our purpose here is to fill in some points that are not discussed completely at the end of Fibonacci's work. Much of the discussion follows pages 115–119 of Sigler's translation and commentary; full citations for the later are given in `history07b.pdf`.

Sigler's symbolic formulation of the problem on page 115 of his translation/commentary is somewhat misstated, so we shall begin with a correct statement of the original problem:

Find (positive rational) numbers x, y, z such that

$$x^2 + x + y + z = u^2 , \qquad u^2 + y^2 = v^2 , \qquad v^2 + z^2 = w^2$$

for suitable postive (rational) numbers u, v, w.

The first step is to find a pair of Pythagorean triples which satisfy the second and third equations, and Fibonacci's choice of triples is $(6, 8, 10)$ and $(10, 24, 26)$. These yield candidates for all the numbers y, z, u, v, w; if we try to solve for x, we find that

$$x = \frac{\sqrt{17} - 1}{2}$$

which of course is irrational, so we have to look more closely. More precisely, we want to find positive rational numbers x and k such that $y = 8k$, $z = 24k$ and

$$x + 8k + 24k + x^2 = (6k)^2 .$$

If we let $R = 6k$, then the preceding equataion reduces to

$$x^2 + x = R^2 - \frac{16}{3} R .$$

One now uses a familiar method employed frequently by mathematicians such as Diophantus and Al-Karaji to eliminate quadratic terms, setting $x = R - a$. If we make this substitution we find that

$$R = \frac{3a(a-1)}{6a - 19} , \qquad x = \frac{a(3a - 16)}{19 - 6a} .$$

Both R and x will be positive provided

$$\frac{19}{6} < a < \frac{16}{3} .$$

If we take $a = 4$, then we obtain Fibonacci's first solution:

$$x = \frac{16}{5} , \qquad y = \frac{48}{5} , \qquad z = \frac{144}{5}$$

Clearly there are infinitely many choices fo a, and these yield infinitely many different solutions.

Fibonacci then proceeds to find an integral solution to the problem. To do this, he switches to a different pair of Pythagorean triples given by $(7, 24, 25)$ and $(25, 60, 65)$. As before we want to find x so that x, $y = 24k$ and $z = 60k$ are (positive) integers satisfying the given conditions. The first of these equations will then have the form $x^2 + x + 24k + 60k = (7k)^2$; once again, there are some misprints on pages 116-117 of Sigler, in which the necessary k-factors are missing (clearly it is impossible to find a positive value of x such that $x^2 + x + 84 = 7^2 = 49$). If we now let $R = 7k$ and recall that $84 = 12 \times 7$, then the crucial equation becomes $x^2 + x + 12R = R^2$, and if we once again write $x = R - a$ we obtain the following:

$$(R - a) \cdot (R - a + 1) \ + \ 12R \ = \ R^2$$

This has the solution

$$R \ = \ \frac{a(a-1)}{2a - 13}$$

and it follows that both R and x will be positive if $a = 7$, in which case $R = 42$ and $k = 6$. It follows that we have the integral solution $x = 35$, $y = 144$ and $z = 360$. This completes the discussion of Master Theodore's problem.

Fibonacci now proceeds to note that one can find a (positive) rational solution to the more complicated system

$$w^2 + w + x + y + z \ = \ r^2 \ , \qquad r^2 + x^2 \ = \ s^2$$

$$s^2 + y^2 \ = \ t^2 \ , \qquad t^2 + z^2 \ = \ u^2$$

where r, s, t, u are suitable (positive) rational numbers. This time one begins with the following three Pythagorean triples:

$$(7, 24, 25) \ , \qquad (25, 60, 65) \ , \qquad (65, 420, 425)$$

We now want to find x and k such that w, k, $x = 24k$, $y = 60k$ and $z = 420k$ solve the given system of equations. In this case the first equation becomes

$$w^2 + w + 504k \ = \ (7k)^2$$

and if we set $R = 7k$ we may rewrite this as

$$w^2 + w + 72R \ = \ R^2 \ .$$

Once again set $w = R - a$, so that

$$R \ = \ \frac{a(a-1)}{2a - 73} \ , \qquad w \ = \ \frac{a(72 - a)}{2a - 73}$$

and observe that both R and w will be positive if

$$\frac{73}{2} \ < \ a \ < \ 72 \ .$$

If we now take $a = 37$ we find the following solution:

$$w \ = \ 1295 \ , \qquad x \ = \ \frac{31968}{7} \ , \quad y \ = \ \frac{79920}{7} \ , \quad z \ = \ 79920$$