Prove that two consecutive odd numbers do not have a common prime factor (in other words, they are relatively prime).

1. Suppose a prime p divides a and b, so $a = pa'$ and $b = pb'$. Then for all integers s and t we know that p divides $sa + tb$, for $sa + tb = p(sa' + tb')$.

2. Given odd integers $\underset{\overset{\shortparallel}{a}}{2k+1}$ and $\underset{\overset{\shortparallel}{b}}{2k+3}$ we know that $2 = b - a$, so if p divides a and b, then p divides 2. Hence the only possible common factor is 2. On the other hand, since $2k+1$ and $2k+3$ are odd we know that 2 divides neither of these numbers. Since we already know that no other prime divides both $2k+1$ and $2k+3$, this means that **no prime** can divide both of them.

Continuation: Find an integer m such that $m \cdot (2k+1) = 1 + n \cdot (2k+3)$. [Some n]

Use long division

$2k+3 = (2k+1) \cdot 1 + 2$ $\qquad a = b \cdot x + y$

$(2k+1) = 2 \cdot k + 1$ $\qquad b = y x_1 + y_1$

Write y_1 as a linear combination of a and b:

$$y_1 = b - y x_1 = b - (a - bx) x_1 = b(x x_1 + 1) - a x_1.$$

Substitute specific values:

$$1 = (2k+1)(1 \cdot k + 1) - (2k+3) \cdot k \stackrel{\text{check}}{=} (2k^2+3k+1) - (2k^2+3k).$$

Therefore if $m = k+1$, then

$$m(2k+1) = 1 + n(2k+3).$$

Specific example — 11 and 13.

$13 = 11 \cdot 1 + 2$ so that

$11 = 2 \cdot 5 + 1$

$$1 = 11 - 2\cdot 5 = 11 - 5\cdot(13 - 11\cdot 1) = 6\cdot 11 - 5\cdot 13 \; [= 66 - 65 = 1].$$

So in this particular case we know that m can be taken to be 6 (there are also other solutions, but it's enough to find one).

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What if we asked for some r such that

$13 \cdot r = 11 s + 1$?

~~Multiply 1 = 6·11 − 5·13 by 11~~:

Use the same equation $1 = 6\cdot 11 - 5\cdot 13$ to conclude that we can let $r = -5$.

Since

$$(6 + 13t)\cdot 11 - (5 + 11t)\cdot 13 =$$

$$6\cdot 11 - 5\cdot 13 = 1,$$ we can also let $r = -5 - 11t$ for any integer t. If we take $t = -1$, this yields $r = 6$.

<u>CHECK</u> $13\cdot 6 = 78 = 1 + 7\cdot 11$.