Math 153
Spring 2010
R. Schultz

EXERCISES RELATED TO `history02.pdf`

As in the earlier exercises, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

- Burton, p. 98: 3, 6, 11*cd*
- Burton, p. 112: 2, 3, 7, 18
- Burton, p. 123: 4
- Burton, p. 478: 3, 6, 11

Additional exercises

0. Suppose that x and y are positive integers such that either both are even or both are odd. Prove that $x^2 - y^2$ is divisible by 4. [*Hint:* The conditions imply that $x - y = 2d$ for some d. Show that $x + y$ is also divisible by 2.]

1. (a) Suppose that m is a perfect number. Prove that

$$1 \;=\; \sum \frac{1}{d}$$

where d runs through all divisors of m except 1 (including $d = m$). [*Hint:* If e runs through all divisors of m except m itself, whey must $d = m/e$ run through all divisors of m except 1?]

(b) Let m be as in the first part of the exercise, and suppose that we are given a rational number r between 0 and 1. Prove that r has an Egyptian fraction expansion

$$r \;=\; \frac{1}{n_1} + \cdots + \frac{1}{n_k}$$

where $n_1 > \cdots > n_k$ and m divides n_k.

2. Let P_n denote the n^{th} pentagonal number. Using the picture in `history02e.pdf` as a starting point, explain why $P_n = P_{n-1} + 3n - 2$, and prove that $P_n = \frac{1}{2}n(3n-1)$.

3. Suppose we have two circles of radius 1 as in the the file `math153exercises02a.pdf`, so that the center of the second circle lies on the first one. Let A denote the area of the lune defined by all points which are inside the first circle but not the second (as the figure suggests, one can reverse "first" and "second" in this discussion). Find A, preferably without using only the standard formula for the area of a region bounded by a circle and the area of a region given by a circular sector of radius r and central angle θ. Some hints accompany the drawing in the indicated file.

4. Using coordinate geometry, prove the following theorem from Aristotle's *Posterior Analytics* which does not appear in Euclid's *Elements*:

> *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be distinct points in the (Euclidean) plane, and let* $0 < r < 1$. *Then the set of all points* $\mathbf{x}$ *such that* $d(\mathbf{p}, \mathbf{x}) = r \cdot d(\mathbf{q}, \mathbf{x})$ *(where "d" denotes the distance between two points) is a circle.*

Here are some suggestions for setting things up. Choose a coordinate system such that the first point is the origin and the second point has coordinates $(a, 0)$ where $a > 0$, and write out the coordinate equation corresponding to the square of the relationship in the statement of the theorem. — This exercises is meant to illustrate the fact that the *Elements* was written to cover only the basics of Greek mathematics at the time and that many known results were not included.