Math 153
Spring 2010
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SOLUTIONS TO EXERCISES FROM `math153exercises07.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 285

6. (*a*) We shall follow the posted hint, starting with the arithmetic progression $a^2 < b^2 < c^2$ with constant difference d. We need to rescale this progression, and the hint is to multiply everything in sight by some $y > 0$. Then the new arithemtic progression has constant difference $y^2 d$, and the middle term is $x^2 = (by)^2$, so that $y = x/b$. We want to choose y so that the common difference is $2x$, then $y^2 d = 2x$. Substituting $x = by$ into this yields $dx^2/b^2 = 2x$, so that $x = d/2b^2$.

7. The condition implies $2n - 1 = u^2$, and we have

$$n^2 \quad = \quad \big((n-1)+1\big)^2 \quad = \quad (n-1)^2 + 2(n-1) + 1 \quad = \quad (n-1)^2 + (2n-1) \quad = \quad (n-1)^2 + u^2 \ .$$

Examples can be given using odd numbers that are perfect squares.■

11. Let x and y be the lengths of the legs of the right triangle, so that $b^2 = \frac{1}{2}xy$ and $a^2 = x^2 + y^2$. Then we have $a^2 + 4b^2 = (x+y)^2$ and $a^2 - 4b^2 = (x-y)^2$. If we solve these for x and y we obtain the values

$$\tfrac{1}{2}\left(\ \sqrt{a^2 + 4b^2} \ \pm \ \sqrt{a^2 - 4b^2}\ \right)$$

as asserted in the problem.■

Problem from Burton, p. 292

2. (*a*) Follow the hint to add the inequalities $F_{2i-1} = F_{2i} - F_{2i-2}$ for $1 \leq i \leq n$. The left hand side will be the sum of the odd Fibonacci numbers from F_1 to F_{2n-1}, and the right hand side is a telescoping sum whose value will be $F_{2n} - F_0 = F_{2n}$.

(*c*) The formula is true for $n = 1$ so proceed by induction and assume it is true for $k \leq n-1$. Then we have

$$\sum_{i=1}^{n} (-1)^{i+1} F_i \quad = \quad \sum_{i=1}^{n-1} (-1)^{i+1} F_i \ + \ (-1)^{n+1} F_n \quad = \quad (-1)^n F_{n-2} + 1 + (-1)^{n+1} F_n$$

and since $F_{n-1} = F_n - F_{n-2}$ the right hand side reduces to

$$1 + (-1)^{n+1} F_{n-1}$$

completing the inductive step of the proof.

3. Let $p = 7, 11, 13, 17$ as in the problem. Here is a short list of Fibonacci numbers and their prime factorization:

$$F_6 = 8, F_8 = 21 = 3 \cdot 7, F_{10} = 55 = 5 \cdot 11, F_{12} = 144 = 2^4 3^2, F_{14} = 377 = 13 \cdot 29$$

$$F_{16} = 987 = 3 \cdot 7 \cdot 47, F_{18} = 2584 = 2^3 \cdot 17 \cdot 19$$

Thus we see that $7|F_8$, $11|F_{10}$, $13|F_{14}$, and $17|F_{18}$.

The site

`http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibtable.html`

contains complete factorization information for the first 300 Fibonacci numbers.

7. (d) We know that $5|n$ if and only if $F_5|F_n$ by the preceding exercises, but $F_5 = 5$.

8. We shall follow the hint in the book. As noted there, the identity $F_n = F_{n+1} - F_{n-1}$ implies that $F_n^2 = F_n(F_{n+1} - F_{n-1})$.

We shall prove the formula by induction on n. If $n = 1, 2$ then one can verify the formula directly, so assume it is true for $n = m$ and consider the case $n = m + 1$. We then have

$$F_1^2 + \cdots + F_m^2 + F_{m+1}^2 = \left(F_1^2 + \cdots + F_m^2\right) + F_{m+1}^2 =$$

$$F_m F_{m+1} + F_{m+1}(F_{m+2} - F_m) = F_{m+1} F_{m+2}$$

proving that Case $(m + 1)$ of the formula is true if Case (m) is true.

SOLUTIONS TO ADDITIONAL EXERCISES

1. Given $y > 0$, define a sequence y_i recursively by $y_1 = y$ and $y_{i+1} = r_i y_i$, andn let a_i be the sequence obtained in this fashion when $r_1 = 1$. Then we have $y_i = a_i y$, and the problem amounts to finding a unique value of y such that

$$x = \sum y_i = \sum a_i y = \left(\sum a_i\right) \cdot y \; .$$

Since $b = \sum a_i$ is a sum of positive numbers, it follows that the desired equation will hold if and only if $x_1 = x/b$, and the remaining x_i are defined recursively by $x_{i+1} = r_i x_i$.

2. We start with the usual geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

which is valid for $|x| < 1$ and differentiate it twice, which we can do by differentiating the summands in a power series (termwise differentiation). This yields

$$\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{2}{(1-x)^3} = \sum_{k=0}^{\infty} (k+2)(k+1)x^k \; .$$

If we multiply the second and third expressions by $\frac{1}{2}$ and set $x = \frac{1}{2}$, then we find that the third expression is just the sum we want while the second is equal to

$$\frac{1}{\left(1 - \frac{1}{2}\right)^3} = 8 \; .$$

3. We shall keep track of the steps symbolically using ordered pairs and the symbols B (boat operator), C (cabbage), G (goat) and W (wolf). Then the initial state is $(BCGW, \text{nothing})$.

At the first step the boat operator has to take the goat, and the result is the state (CW, BG). Next, the boat operator must go back with nothing or else everything is the same as it was at the beginning. Thus after the second step we have (BCW, G). Now the boat operator can take either the wolf or cabbage with him on the next trip, so we shall arbitrarily choose the cabbage, which leads to the new state of (W, BCG). The only option now is to take the goat back, yielding the state (BGW, C). Again there is no choice, and the boat operator must take the wolf, which yields the state (G, BWC). Now the boat operator can return with nothing, so that the resulting state is (BG, WC), and finally the operator can take the goat back to reach the desired state $(\text{nothing}, BCGW)$.

The logic of this problem is symmetric in "cabbage" and "wolf" (neither can be left alone with the goat), and if one switches these in the discussion another valid solution to the problem is obtained (recall we made an arbitrary choice at the third step).