Math 153
Spring 2010
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SOLUTIONS TO EXERCISES FROM `math153exercises11.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 361

7. The defining equation for the relation $L = \text{Naplog } N$ is

$$N = 10^7 \cdot (1 - 10^{-7})^L .$$

For the first part of the problem let's replace M and N by N_1 and N_2. Then we have

$$\frac{N_1}{N_2} = \frac{10^7 \cdot (1 - 10^{-7})^{L_1}}{10^7 \cdot (1 - 10^{-7})^{L_2}}$$

so that

$$N_1/N_2 = (1 - 10^{-7})^{L_1 - L_2} .$$

The number $K = \text{Naplog } 1$ satisfies the equation

$$1 = 10^7 \cdot (1 - 10^{-7})^K$$

so we can rewrite the expression for N_1/N_1 as

$$N_1/N_2 = 1 \cdot (1 - 10^{-7})^{L_1 - L_2} = 10^7 \cdot (1 - 10^{-7})^K \cdot (1 - 10^{-7})^{L_1 - L_2} =$$

$$10^7 (1 - 10^{-7})^{K + L_1 - L_2}$$

so that $\text{Naplog } N_1/N_2 = K + L_1 - L_2 = \text{Naplog } N_1 - \text{Naplog } N_2 + \text{Naplog } 1$, which is what we wanted to show.■

To prove the second part, we use the identity

$$\text{Naplog } N_1 N_2 = \text{Naplog } N_1 + \text{Naplog } N_2 - \text{Naplog } 1$$

in the eighth line on page 354 of Burton along with induction on the exponent r.■

8. We are given the following sequence of assertions:

$$\text{Naplog } N = \log_{(1-10^{-7})} (10^{-7} N) =$$

$$10^7 \log_{(1-10^{-7})^{(10^{-7})}} (10^{-7} N) =$$

$$10^7 \log_{1/e}(10^{-7} N) \quad = \quad 10^7 \log_e(N/10^7) \ .$$

The formula on the first line follows from the equation at the top of this page (divide both sides by 10^7).

The second expression on the first line is equal to the expression on the second line because of the identity

$$\log_b \ x \quad = \quad \frac{1}{a} \log_{b^a} x$$

(if $x = b^c$ then $x = (b^a)^{(c/a)}$).

The expression on the second line is **approximately** equal to the first expression on the third line because

$$\frac{1}{e} \quad = \quad \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$$

and if we take $n = 10^7$ the approximation is already extremely good (alternatively, one can use the Chain Rule for logarithms to express the approximation more precisely).

The final two terms are equal because of the following equations:

$$\log_{1/b}(u/v) \quad = \quad -\ \log_b(u/v) \quad , \quad -\log_b(u/v) \quad = \quad -\ \log_b(v/u)$$

Combining these observations yields the assertion that the Napier logarithm is essentially equivalent to the natural logarithm; in fact, we have a formula

$$\text{Naplog } N \quad = \quad 10^7(p - q \ \log_e \ N)$$

where q is very close to 1.■

NOTE. One can use the formula in Exercise 10 (or a modification of it for $\log_e$ rather than $\log_{10}$ to obtain simple formulas for the derivative and indefinite integral of the Napier logarithm function.■