Math 153
Spring 2010
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SOLUTIONS TO EXERCISES FROM `math153exercises11a.pdf`

As usual, "Burton" refers to the Seventh Edition of the course text by Burton (the page numbers for the Sixth Edition may be off slightly).

Problems from Burton, p. 380

4. If we multiply the polynomial by $(x+1)$ we obtain $x^4 + x^2 + 3x + 1$, and by Descartes' Rule of Signs this polynomial has no positive real roots. Therefore the same is try for any factor of this polynomial, so in particular it is true for $p(x) = x^3 - x^2 + 2x + 1$. Since the constant term is nonzero, it also follows that 0 is not a root of $p(x)$. Now $p(-x) = -x^3 - x^2 - 2x + 1$, so again by Descartes' Rule of Signs this polynomial has exactly one negative root. It follows that $p(x)$ factors into a product of a first degree polynomial with a negative real root and a second degree polynomial with no real roots.

7. (b) Let $p(x)$ be the given polynomial. Then both $p(x)$ and $p(-x)$ have exactly two sign changes, and $p(0) \neq 0$, so $p(x)$ has at most four real roots and hence must have at least two nonreal complex roots.

8. (b) Since only odd powers of the original polynomial are nonzero, the constant term is zero and hence we can write the polynomial in the form $p(x) = x \cdot f(x)$, where $f(x)$ is a nonzero polynomial with only terms in even degrees. By our assumptions the coefficients of $f(x)$ are all nonnegative and at least one is positive. This means that $f(x)$ has no positive real roots, which in turn implies the same for $p(x)$. On the other hand, there are also no sign changes for $p(-x) = (-x)\, f(-x)$, for the latter only has nonzero terms in odd degrees and the nonzero coefficients must all be negative. Therefore $p(x)$ has no negative roots. Since we know that 0 is a root because x divides $p(x)$, it follows that 0 is the unique real root of $p(x)$. — Note that 0 may well be a repeated root of such a polynomial, as it is in the case of $p(x) = x^5 + x^3$.

(c) There are no sign changes in the polynomial under consideration.

9. (a) Since every odd degree polynomial has at least one real root, it will suffice to show that there are no roots which are nonnegative. But since the constant term is nonzero it follows that 0 is not a root, and by Descartes' Rule of Signs we also know that there are no positive roots. Hence there must be exactly one negative root and two additional nonreal roots (which are complex conjugates of each other by the Quadratic Formula).

(b) Consider the polynomial $p(x) = x^3 - a^2x + b^2$. Since $p(0) > 0$ it follows that there must be at least one negative root.

(c) We are given $p(x) = x^4 + a^2x^2 + b^2x - c^2$ where $c \neq 0$. There is one sign change in $p(x)$, so by Descartes' Rule of Signs there must be exactly one positive real root. Since $c^2 > 0$ it follows that 0 is not a root. Finally, since $p(-x) = x^4 + a^2x^2 - b^2x - c^2 = 0$ there is also one sign change in $p(-x)$ and therefore we must also have exactly one negative root. This means that $p(x)$ must be the product of a polynomial with one positive root, one negative root, and two nonreal roots.

Alternate approach using calculus. To simplify the discussion we shall assume that a and b are both nonzero, so that $a^2, b^2 > 0$; one can treat the cases $a = 0$ or $b = 0$ similarly. — Since $p(x)$ is a quartic polynomial with leading term x^4, we have

$$\lim_{x \to \pm\infty} p(x) \quad = \quad +\infty$$

so that $p(0) = -c^2 < 0$ implies there is at least one positive root and one negative root. Since

$$p'(x) \quad = \quad 4x^3 \ + \ 2a^2 x \ + \ b^2$$

we know that $p'(x) > 0$ for $x > 0$ and hence p is strictly increasing for $x \geq 0$, so that there can be only one positive real root. As before, 0 is not a root, so the only options are that $p(x)$ has one negative root or three negative roots (counted with multiplicities). We need to eliminate the second possibility. — Since $p''(x) = 12x^3 + 2a^2$, it follows that $p''(x) > 0$ everywhere so that the third degree polynomial $p'(x)$ is strictly increasing and hence has exactly one real root. Suppose there are at least two distinct negative roots of p, so that with the positive root we have $r_2 < r_1 < 0 < r_0$. Then by Rolle's Theorem we must also have $p'(x) = 0$ for some values of x between r_2 and r_1 and also between r_1 and r_0. But we have just seen this is impossible. So the only remaining alternative is that $p(x)$ has repeated roots. If this happens, then p and p' have a common root. Since $p'(x) > 0$ for $x \geq 0$ it follows that a positive root cannot be a repeated root. This leaves us with only one possibility; namely, there is a single negative root which is a triple root. In other words, our original polynomial $p(x)$ must have the form $(x - r)\,(x + s)^3$ for some $r, s > 0$. But if this is the case, then it will follow that s is also a root of $p''(x)$, and we know this polynomial has no real roots whatsoever. Hence there can only be one positive real root, one negative real root, and two nonreal roots (which again are conjugate to each other).