1. Mathematics in the earliest civilizations

(Burton, **1.1-1.3**, **2.2 – 2.5**)

Both archaeology and anthropology show that most if not all human cultures have had at least some crude concepts of numbers, with the earliest archaeological evidence scientifically dated around 30,000 years ago. Numerous archaeological discoveries also indicate that numerous prehistoric cultures had discovered that counting larger quantities was easier with some means of grouping together fixed numbers of objects. For example, twelve stones could be arranged in two groups of five and one group of two, and similarly for other numbers one can count off groups of five until there are less than five items left. Such arrangements are the first step in the development of a number system.

Although the study of rudimentary number concepts in prehistoric and other primitive cultures is potentially an interesting subject, for our purposes it will be best simply to recognize the near-universal awareness of the number concept as given and to move on to the development of mathematics within the ancient civilizations that emerged about 5000 years ago. In this unit we shall focus on two civilizations that have had a particularly strong impact on mathematics as we know it today; namely, Egyptian and Mesopotamian civilizations. Extensive information on numbering systems in other cultures is contained in the following reference:

> Ifrah, Georges,. The universal history of numbers. From prehistory to the invention of the computer*..* Translated from the 1994 French original by David Bellos, E. F. Harding, Sophie Wood and Ian Monk. *John Wiley & Sons, Inc., New York,* 2000. ISBN: 0–471–37568–3**.**

Existing records of ancient civilizations are often very spotty in many respects, with very substantial information on some matters and little if anything on others. Therefore any attempt to discuss mathematics in ancient civilizations must recognize that one can only discuss what is known from currently existing evidence and accept that much of these cultures' mathematics has been lost with the passage of time. However, we can safely conclude that such cultures were quite proficient in some aspects of practical mathematics, for otherwise many of the spectacular engineering achievements of ancient cultures would have been difficult to plan and impossible to complete. All statements made about pre-Greek mathematics must be viewed in this light, and for civilizations in other parts of the world (China in particular) the unevenness of evidence extends to even later periods of time.

Expressions for whole numbers in Egypt and Mesopotamia

In most but not all cases, the development of written records is closely linked to the birth of a civilization, and many such records are basically numerical. Therefore we have some understanding of the sorts of numbering systems used by most of the ancient civilizations. In most cases it is apparent that these civilizations had also discovered the concept of fractions and had devised methods for expressing them.

One extremely noteworthy point is that different civilizations often took quite different approaches to the problem of setting up workable number systems, and this applies particularly to fractions. Perhaps the simplest question about number systems concerns the choices for grouping numbers. The numbers 5 and 10 usually had some particular significance. For example, Egyptian hieratic writing had separate symbols for 1 through 9, multiples of 10 up to 90, multiples of 100 through 900, and multiples of 1000 through 9000; it should also be noted that the earlier hieroglyphic Egyptian writing included symbols for powers of 10 up to ten million). A number like 256 would then be represented by the symbols for 200, 50 and 6; this is totally analogous to the Roman numeral expression for 256 as CCLVI. Numerous other cultures had similar systems; in particular, in classical Greek civilization the Greek language used letters of the alphabet to denote numbers from 1 to 9, 10 to 90 and 100 to 900 in exactly the same fashion.

Although 10 has played a key role in most number systems, there have been some notable exceptions, and traces of some are still highly visible in today's world. The Mayan civilization placed particular emphasis on the numbers 5 and 20. Roman numerals indicate a special role for 5 and 10. However, the Sumerians in Mesopotamia developed the most extraordinary alternative during the third millennium B. C. E. They used a ***sexagesimal*** (or base 60) system that we still use today for telling time and some angle measurements: One degree or hour has sixty minutes, and one minute has sixty seconds. The Mesopotamian notation for numbers from 1 to 59 is strikingly similar to the notation we use today. In particular, if n is a positive integer less than 60 and we write

$$n = 10p + q \quad \text{where} \quad 0 \le p \le 5 \text{ and } 1 \le q \le 9$$

then n was written as a combination of p thick horizontal strokes and q thin vertical strokes. Much like our modern number system, larger positive integers were expressed in a form like

$$a_0 + a_1 \times 60 + a_2 \times 60^2 + a_3 \times 60^3 + \ldots + a_N \times 60^N$$

where each a_j is a nonnegative integer that is less than 60, but at first there were problems when one or more of the numbers a_j was equal to zero, and eventually place holders were used in positions where we would insert a zero today.

However, as noted on page 23 of Burton, there is nothing to indicate that any such place holder was "regarded … as a number by itself that could ever be used for computational purposes."

Egyptian fractions

The differences between the Egyptian and Mesopotamian representations for fractions were far more significant. For reasons that are not really understood, the Egyptians expressed virtually all fractions as finite sums of ordinary reciprocals or *unit fractions* of the form

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + (etc.)$$

where ***no denominator appears more than once in the expansion***. Of course this restriction leads to complicated expressions even for many fairly simple fractions, and the discussion and tables on pages 37–38 of Burton give expansions for a long list of fractions with small denominators.

As noted on page 41 of Burton, every rational number between 0 and 1 has an ***Egyptian fraction*** expansion of this type. Perhaps the most widely known method for finding such expressions is the so-called ***greedy algorithm*** due to the thirteenth century Italian mathematician Leonardo of Pisa (better known as ***Fibonacci***). A description of this method and a proof that it works are reproduced below, this account is slightly adapted from the online site

<http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fractions/egyptian.html#Fibgreedy>

which also has information on several other aspects of Egyptian fractions.

Before proving the general result on Egyptian fraction expansions, it seems worthwhile to make some general comments. By the early thirteenth century mathematics had outgrown the need for Egyptian fraction expansions to do arithmetic with fractions. However, there was enough remaining interest for a leading mathematician of the day to give a logically rigorous procedure for finding such expansions. Thus a problem originally arose in a "practical" context had interesting features that generated further study of the general topic for their own sake. This happens frequently in many areas of human activity, but it is particularly fundamental to mathematics; it was already apparent in the Rhind papyrus from the nineteenth century B. C. E. The reasons for pursuing such mathematical questions for their own sake frequently go beyond simple curiosity and enjoyment. Very often a topic that originally seemed interesting in itself eventually figures in a serious mathematical inquiry. Further discussion of this appears in an article by David Singmaster (*The unreasonable utility of recreational mathematics*), which is available at the following online site:

http://anduin.eldar.org/~problemi/singmast/ecmutil.html

As an indication of how Egyptian fraction expansions have continued to generate mathematical interest we mention an unsolved problem raised by the celebrated mathematician Paul Erdős (1913–1996) and E. G. Straus**:** *Suppose that n is an odd number which is greater than or equal to* 5. *Is it always possible to write the fraction* $^{4}/_{n}$ *as a sum of three unit fractions?*

Some results on Egyptian fraction expansions that are related to this problem are discussed in an addendum to this section; *i.e.*, document **history01a.pdf** (or the alternate version **history01a.ps**), which is available in the course directory **:**

http://math.ucr.edu/~res/math153

Finding Egyptian fraction expansions. We now return to a statement and proof of the Greedy Algorithm for expressing an arbitrary fraction as a sum of unit fractions.

FIBONACCI'S METHOD A. K. A. THE GREEDY ALGORITHM: This method and a proof are given by Fibonacci in his book *Liber Abaci* produced in 1202 that introduced the rabbit problem involving the Fibonacci Numbers. We begin by noting that

- $T / B < 1$ and
- if $T = 1$ the problem is solved since T / B is already a unit fraction, so
- we are interested in those fractions where $T > 1$.

The method is to find the ***biggest unit fraction*** we can and take away from T / B and hence the other name for this process – the ***Greedy Algorithm***.

With what is left, we repeat the process. We will show that this series of unit fractions always decreases, never repeats a fraction and eventually will stop. Such processes are now called *algorithms* and this is an example of a *greedy algorithm* since we (greedily) take the largest unit fraction we can and then repeat on the remainder.

Let's look at an example before we present the proof:

$$^{521}/_{1050}.$$

Now $^{521}/_{1050}$ is less than one-half (since 521 is less than a half of 1050) but it is bigger than one-third. So the largest unit fraction we can take away from $^{521}/_{1050}$ is $^{1}/_{3}$:

$$^{521}/_{1050} \;=\; ^{1}/_{3} + R$$

What is the remainder? To find it we simply subtract one fraction from the other:

$$^{521}/_{1050} \quad - \quad ^{1}/_{3} \quad = \quad ^{57}/_{350}$$

So we repeat the process on $^{57}/_{350}$:

This time the largest unit fraction less than $^{57}/_{350}$ is $^{1}/_{7}$ and the remainder is $^{1}/_{50}$.

> How do we know it is 7? Divide the bottom (larger) number, 350, by the top one, 57, and we get 6.14 **...** . So we need a number larger than 6 (since we have 6 + 0.14 ...) and the next one above 6 is 7.

So $^{521}/_{1050} = {}^{1}/_{3} + {}^{1}/_{7} + {}^{1}/_{50}$. The sequence of remainders is important in the proof that we do not have to keep on doing this for ever for some fractions T / B:

$$^{521}/_{1050}, \quad ^{57}/_{350}, \quad ^{1}/_{50}$$

In particular, although the *denominators* of the remainders are getting bigger, the important fact that is true in *all cases* is that *the numerator of the remainder is getting smaller*. If it keeps decreasing then it must eventually reach 1 and the process stops.

A PROOF: Now let's see how we can show this is true for all fractions T / B. We want

$$T / B \quad = \quad 1/u_1 \ + \ 1/u_2 \ + \ \ldots \ + \ 1/u_n$$

where $u_1 < u_2 < \ldots < u_n$. Also, we are choosing the largest u_1 at each stage.

What does this mean? It means that $1/u_1 < T / B$, but also that $1/u_1$ is the *largest* such fraction. For instance, we found that $^{1}/_{3}$ was the largest unit fraction less than $^{521}/_{1050}$. This means that $^{1}/_{2}$ would be *bigger* than $^{521}/_{1050}$.

In general, if $1/u_1$ is the largest unit fraction less than T / B then

$$1/(u_1 - 1) \quad > \quad T / B.$$

Since $T > 1$, neither $1/u_1$ nor $1/(u_1 - 1)$ is equal to T / B. What is the remainder? It is

$$T / B \ - \ (1/u_1) \quad = \quad (T \cdot u_1 \ - \ B) / (B \cdot u_1)$$

Also, since $1/(u_1 - 1) > T/ B$, then multiplying both sides by B we have

$$B / (u_1 - 1) \quad > \quad T$$

or, multiplying both sides by $(u_1 - 1)$ and expanding the brackets, then adding T and subtracting B to both sides we have:

$$B > T \cdot (u_1 - 1)$$
$$B > T \cdot u_1 - T$$
$$T > T \cdot u_1 - B$$

Now $T \cdot u_1 - B$ was the *numerator of the remainder* and we have just shown that *it is smaller than the original numerator* T. If the remainder, in its lowest terms, has a 1 on the top, we are finished. Otherwise, we can *repeat the process* on the remainder, which has a smaller denominator and so the remainder when we take off its largest unit fraction gets smaller still. Since T is a whole (positive) number, this process *must* inevitably terminate with a numerator of 1 at some stage.

This completes the proof of the following statements:

- There is always a *finite* list of unit fractions whose sum is any given fraction T / B
- We can find such a sum by taking the largest unit fraction at each stage and repeating on the remainder (the *greedy algorithm*)
- The unit fractions so chosen get smaller and smaller (and so all are unique)

DIFFERENT REPRESENTATIONS FOR THE SAME FRACTION: We obviously have $^3/_4 = {}^1/_2 + {}^1/_4$, but there are also other Egyptian fraction forms for $^3/_4$. For example we have $^3/_4$ as $^1/_2 + {}^1/_5 + {}^1/_{20}$ and $^1/_2 + {}^1/_6 + {}^1/_{12}$ and $^1/_2 + {}^1/_7 + {}^1/_{14} + {}^1/_{28}$. One could continue in this manner, but instead of doing so we shall discuss the underlying general principle:

EACH FRACTION BETWEEN 0 AND 1 HAS AN INFINITELY MANY EGYPTIAN FRACTION REPRESENTATIONS: We begin with an absolutely trivial observation:

$$1 = {}^1/_2 + {}^1/_3 + {}^1/_6$$

By the Greedy Algorithm we know that every fraction T/B as above has at least one Egyptian fraction expression. This will be the first step of a proof by induction. Suppose that we have k distinct expressions as an Egyptian fraction for some positive integer k. To complete the inductive step we need to construct one more expression of this type for T/B.

Among all the k given Egyptian fraction expansions there is a minimal unit fraction summand 1/m. Choose one of these expansions

$$T / B = 1/u_1 + 1/u_2 + \ldots + 1/u_n$$

such that $u_n = m$. Then we may use the trivial identity above to obtain the following equation:

$$T / B \;=\; 1/u_1 + 1/u_2 + \ldots + 1/u_{n-1} + 1/(2 \cdot u_n) + 1/(3 \cdot u_n) + 1/(6 \cdot u_n)$$

We claim this expression is different from the each of k expressions that we started with. To see this, observe that the minimal unit fraction summand is equal to

$$1/(6 \cdot u_n) \;=\; 1/(6 \cdot m)$$

but the minimal unit fraction summand for each expression on the original list is at least $1/m$. Therefore we have obtained an expression that is not on the original list of k equations, which means there are at least k+1 different expressions for T / B and thus completes the proof of the inductive step.

In contrast, one can also prove that for a fixed positive integer L there are only finitely many ways of expressing T / B as a sum of at most unit fractions. The proof of this by mathematical induction will be left to the exercises.

Babylonian fractions

We begin with some standard terminology that may be misleading in some respects but is generally used and convenient. Mesopotamia had a succession of dominant kingdoms from the dawn of civilization through the Persian conquest in 538 B. C. E. In discussions of the ancient mathematical history for this region, it is customary to use the term "Babylonian mathematics" for contributions during the period from approximately 2000 B. C. E. through the Persian conquest.

As one might guess from the present day hierarchies of hours/minutes/seconds and degrees/minutes/seconds, Babylonian mathematics extended its sexagesimal numbering system to include representations for fractions. This crucial step gave Babylonian mathematics a huge computational advantage over Egyptian mathematics and allowed the Babylonians to make extremely accurate computations; their methods and results were unsurpassed by other civilizations until the Renaissance in Europe, nearly nineteen centuries after the Persian conquest.

Of course, it is not possible to express every rational number between 0 and 1 as a finitely terminating sexagesimal fraction. For example, the formula for $^1/_7$ in sexagesimal form is

$$0;8,34,17,8,34,17,8,34,17,8,34,17,8,34,17,8,34,17,\underline{8,34,17,}$$

where the underlining indicates an infinitely repeating periodic sequence of the given numbers. Babylonian mathematics almost always ignored such expressions, and they are conspicuously absent from the tables of fractions that are known to exist.

Mathematical legacies of the early civilizations

Both the Egyptians and Babylonians were quite proficient in using arithmetic to solve everyday problems. Furthermore, both civilizations complied substantial tables of values to be used when working problems. However, the emphasis was on specific problems rather than general principles. In particular, there is no evidence of proofs or comprehensive explanations of computational procedures, and the general discussions of procedures seemed to be directed at facilitating techniques rather than developing understanding. Given such an empirical approach, it was probably inevitable that there are mistakes in some of their procedures for finding answers to specific types of problems. Chapter 2 of Burton mentions one specific formula that both cultures got wrong: Given a "nice" quadrilateral ABCD in the plane such that the lengths of sides AB, BC, CD and DA are a, b, c and d respectively, the writings of both cultures give the following formula for the area enclosed by ABCD:

$$\textbf{area} \quad \text{is supposedly equal to} \quad {}^{1}/_{4}\,(\mathbf{a} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{d})$$

Further discussion of this estimate for the area appears in Exercise 8 on page 56 of Burton (see also Exercise 5 on page 75). In a few cases, one culture found the right formula for computing something while the other did not. One example of this sort involves the volume of a frustrum of a pyramid with a square base; this is the object formed by taking a pyramid with a square base and slicing off the top along a plane that is parallel to the base; the Egyptian formula was correct but the Babylonian one was not. An excellent interactive graphic for this figure and some interesting commentary on the Egyptian formula are available online at the site

http://mathworld.wolfram.com/TruncatedSquarePyramid.html

and the related site for the pyramidal frustrum listed there is also worth viewing.

Both the Egyptian and Mesopotamian civilizations developed numeration systems (with fractions) that were highly adequate for their purposes in many respects, but in each case there were difficulties with their approaches to fractions. In Egyptian mathematics, the most obvious problem concerned the clumsiness of the manner in which they wrote fractions, while in Babylonian mathematics there was the problem of dealing accurately with fractions T/B for which the reduced version's (*i.e.*, T and B have no common factors except +1 and –1) denominator B is divisible by a prime greater than 5. More generally, the general lack of clear distinctions between approximate and actual values was also a problem for both Egyptian and Babylonian mathematics.

Achievements and weaknesses of Egyptian mathematics

The Egyptian civilization was the first known to develop systematic calendars based upon lunar and solar cycles, maybe as early as the fifth millennium B. C. E. Some of their numerical estimation procedures were quite good and elaborations of a few are still used today for some purposes (*e.g.*, the rule of false position), and Egyptian mathematics was clearly able to approximate square roots effectively. The existing documents and monuments all indicate a more extensive understanding of geometry than in Babylonian civilization. In particular, as noted before the Egyptians knew how to compute the volume of a truncated pyramid (see the bottom of page 52 in Burton) but the Babylonian formula was incorrect. Although it is clear that Egyptian geometry provided valuable input to the later work of Greek geometers, evidence also suggests that the extent of these contributions was substantially less than classical Greek writers like Herodotus indicated.

Egyptian arithmetic was based very strongly on addition and subtraction, and both multiplication and division were carried out by relatively awkward additive procedures. For example, if one wanted to multiply two positive integers A and B, this would begin with adding A to itself to form 2 A, adding 2 A to itself to form 4 A, and so on until one reaches the largest power of 2 such that $2^k \leq$ B. The next step would amount to finding the base two expansion of B as

$$1 + 2 + \ldots + 2^k$$

and the final step would be to add all the numbers 2^p A such that 2^p is a term which appears in the binary expansion of B. Of course, computing this way is extremely clumsy by today's standards, but clearly the Egyptians were able to live with this and use it very successfully for many purposes.

One particularly noteworthy feature of Egyptian mathematics is that it apparently changed very little over a period of approximately two thousand years.

Achievements and weaknesses of Babylonian mathematics

The sexagesimal numeration system provided an extremely solid foundation for doing all sorts of calculations to a very high degree of accuracy, and Babylonian mathematics realized this potential with an extensive collection of algorithms. In particular, their method for solving quadratic equations is equivalent to the quadratic formula that we still use in many situations. Babylonian mathematics was also quite proficient in solving large classes of cubic equations and systems of two equations in two unknowns. There were also other achievements related to algebra, but here we shall only mention the existence of evidence that

Babylonian mathematics had some primitive understanding of trigonometric, exponential and logarithmic functions (however, we should stress that the extent of this understanding was extremely limited).

Although it appears that Egyptian mathematics understood at least some important special cases of the Pythagorean Theorem, the first known recognition of the Pythagorean Formula appears in Babylonian mathematics; in keeping with our earlier comments, there ware no attempts to justify the formula. However, the Babylonians also studied integral solutions of the Pythagorean Equation

$$a^2 + b^2 = c^2$$

extensively; *e.g.*, this is evident from the cuneiform tablet called Plimpton 322.

We have mentioned evidence that Egyptian geometry was relatively well–developed. It is more difficult to assess Babylonian achievements in this area, but the evidence does show a high level of proficiency in making geometric measurements of various kinds.

Babylonian mathematics had a particularly strong impact on observational astronomy, including the traditional notation for locating heavenly bodies and prediction of solar and lunar eclipses.

Some weaknesses of Babylonian mathematics were already mentioned above. The lack of negative numbers is also worth mentioning, both for its own sake and its relation to another deficiency: Babylonian mathematics only described one solution for a quadratic equation rather than two. Of course, even if one ignores negative numbers there are many cases for which a quadratic equation has two positive roots. Given that high school level second year courses in algebra often contain exercises to illustrate the pitfalls of equations involving square roots, recognizing only one root of a quadratic equations can clearly be a nontrivial problem in some situations.

We conclude this unit with an important caution about the preceding discussion. Although there is no direct evidence of proofs or comprehensive explanations of computational procedures, describing mathematics in early civilizations as a purely utilitarian subject – which was devoid of logical structure, deductive proofs, generalizations or abstractions – is probably an oversimplification. Certainly none of these features appeared as explicitly or forcefully as they do in Greek mathematics, but the evidence also suggests some forms of these features must have been at least implicit. Among the arguments suggesting this are (1) descriptions of numerous problems that resemble each other quite closely and solving them by similar methods, (2) performing arithmetic operations with different types of measurements – for example, adding a length to an area. As noted before, our information on Egyptian and Mesopotamian mathematics is extremely spotty and there is plenty of room for speculation in many directions.