1.A. Egyptian fractions with small denominators

In Unit 1 we mentioned P. Erdős's question about expressing a fraction of the form $4/n$, where n is odd, as a sum of at most three unit fractions with different denominators. The purpose of this writeup is to put this result into perspective by making three closely related observations:

FACT I. *If n is odd, then $2/n$ can be expressed as a sum of two unit fractions with different denominators.*

FACT II. *If n is not divisible by 3, then $3/n$ can be expressed as a sum of at most three unit fractions with distinct denominators. Furthermore. there are fractions of this type than cannot be expressed as a sum of two unit fractions with distinct denominators.*

FACT III. *If n is odd, then $4/n$ can be expressed as a sum of at most four unit fractions with distinct denominators. Furthermore. there are fractions of this type than cannot be expressed as a sum of two unit fractions with distinct denominators.*

We shall prove these results in the order stated.

Fractions with numerators equal to 2

Suppose that the numerator is 2. If the denominator is even then we have

$$\frac{2}{2s} \quad = \quad \frac{1}{s}$$

so this case is not particularly interesting. This is why we assume that the denominator is odd. Denote the latter by $2r+1$. If we apply the Greedy Algorithm we see that $\frac{1}{r+1}$ is the largest unit fraction $\leq \frac{2}{2r+1}$. But now we have

$$\frac{2}{2r+1} \;-\; \frac{1}{r+1} \quad = \quad \frac{1}{(r+1)\,(2r+1)}$$

and if we add $\frac{1}{r+1}$ to both sides we obtain the an expression for $\frac{2}{2r+1}$ as a sum of two unit fractions. To complete the proof of Fact I when the denominantor is 2, we need to show that the denominators are distinct; *i.e.*, we need to check that

$$r+1 \quad \neq \quad (2r+1)\cdot(r+1)\ .$$

In fact, since $2r+1 \geq 3$ the right hand side is at least 3 times the left hand side, so the denominators are clearly distinct as required.∎

Fractions with numerators equal to 3

As in the previous discussion, we might as well assume that the denominator is not divisible by 3. There are two cases depending upon whether the denominator has the form $3r+1$ for some $r \geq 1$ or $3r+2$ for some $r \geq 0$.

Suppose first that the denominator has the form $3r+2$ for some $r \geq 0$. In this case we can imitate the previous argument to show that

$$\frac{3}{3r+2} \;-\; \frac{1}{r+1} \quad = \quad \frac{1}{(r+1)\,(3r+2)}$$

and as before this yields an Egyptian fraction expansion of $\frac{3}{3r+2}$ if and only if the denominators $r+1$ and $(r+1)\cdot(3r+2)$ are distinct. This follows because $3r+2 \geq 2$.

Suppose now that the denominator has the form $3r+1$ for some $r \geq 1$. In this case the same sort of calculation yields

$$\frac{3}{3r+1} \quad = \quad \frac{2}{(r+1)\,(3r+1)} \quad + \quad \frac{1}{(r+1)}$$

and now there are two cases depending upon whether the expression $(r+1)\cdot(3r+1)$ is even or odd. Easy calculations show that the product is even if r is odd and odd if r is even.

Both cases will be easier to handle if we first show that

$$\frac{2}{(r+1)\,(3r+1)} \quad < \quad \frac{1}{(r+1)}$$

for all $r \geq 1$. Simple manipulation shows that the latter is equivalent to the inequality

$$(r+1)\,(3r+1) \quad > \quad 2\,(r+1)$$

and the latter is clearly true since $r \geq 1 \Longrightarrow 3r+1 \geq 4 > 2$.

Suppose now that $r = 2s-1$ is odd (since $r \geq 1$ we must have $s \geq 1$). Then we have

$$\frac{2}{(r+1)\,(3r+1)} \quad = \quad \frac{2}{(2s)\,(6s-2)} \quad = \quad \frac{1}{s\,(6s-2)}$$

and by the preceding inequality we know that this is less than $1/r+1$. Therefore we have the identity

$$\frac{3}{3r+1} \quad = \quad \frac{1}{s\,(6s-2)} \quad + \quad \frac{1}{(r+1)}$$

where the two summands on the right are distince by the remarks of the previous paragraph. Therefore if r is odd there is an Egyptian fraction expansion of $\frac{3}{3r+1}$ as a sum of two unit fractions with different denominators.

Suppose now that r is even, so that $r = 2s$ for some $s \geq 2$. Then by the preceding calculations we may write

$$\frac{2}{(2s+1)\,(6s+1)} \quad = \quad \frac{2}{12s^2+8s+1} \quad = \quad \frac{1}{6s^2+4s+1} \quad + \quad \frac{1}{(6s^2+4s+1)\,(12s^2+8s+1)}$$

and therefore our original fraction $\frac{3}{3r+1}$ will be a sum of three unit fractions if all the denominators are distinct. By our results on expressing $\frac{2}{2m+1}$ as a sum of two unit fractions, we know that the denominators $6s^2+4s+1$ and $(6s^2+4s+1)\,(12s^2+8s+1)$ are distinct. To finish work on this case we need to show that neither of these is equal to $r+1$. But the previously established inequality shows that

$$\frac{1}{r+1} \quad > \quad \frac{2}{(2s+1)\,(6s+1)} \quad = \quad \frac{1}{6s^2+4s+1} \quad + \quad \frac{1}{(6s^2+4s+1)\,(12s^2+8s+1)}$$

and thus the unit fraction $1/r+1$ is greater than either of the other two summands in the expansion of $\frac{3}{3r+1}$. This means that the latter has been expressed as a sum of three distinct unit fractions as required.

To conclude the discussion of the cases where the denominator is equal to 3, we need to show that some fractions of the form $\frac{3}{3r+1}$ cannot be written as a sum of two unit fractions with distinct denominators. Suppose that $r = 2$ so that the fraction in question is $\frac{3}{7}$ If we can write

$$\frac{3}{7} \quad = \quad \frac{1}{a} + \frac{1}{b}$$

where $a \neq b$, then either $a > b$ or vice versa. We shall assume $a < b$; the other case follows by reversing the roles of a and b. We then have

$$\frac{1}{a} + \frac{1}{a} \quad > \quad \frac{1}{a} + \frac{1}{b} \quad = \quad \frac{3}{7} \quad > \quad \frac{1}{a}$$

which we can rewrite in the form

$$\frac{2}{a} \quad > \quad \frac{3}{7} \quad > \quad \frac{1}{a}$$

and since a is an integer the preceding inequalities imply

$$3 \quad \leq \quad a \leq \quad 4 \ .$$

The proof that $\frac{3}{7}$ does not have an Egyptian fraction expansion with two terms thus reduces to the following computations:

$$\frac{3}{7} - \frac{1}{3} \quad = \quad \frac{2}{21} \quad , \quad \frac{3}{7} - \frac{1}{4} \quad = \quad \frac{5}{28}$$

Since neither term on the right hand side is a unit fraction, the assertion regarding expansions of $\frac{3}{7}$ follows immediately.■

Fractions with numerators equal to 4

Once again, we might as well assume the denominator is odd. If it is divisible by 4 then $4/4s$ is a unit fraction, and if it has the form $4r + 2$, then

$$\frac{4}{4r+2} \quad = \quad \frac{2}{2r+1}$$

and consequently it has an Egyptian fraction expansion as a sum of two distinct unit fractions.

Once again there are two cases depending upon whether the denominator has the form $4r + 1$ or $4r + 3$; since we are working with fractions in the unit interval we must have $r \geq 1$. If the denominator has the form $4r + 1$, then we may imitate the previous construction to write

$$\frac{4}{4r+3} - \frac{1}{r+1} \quad = \quad \frac{1}{(r+1)\,(4r+3)}$$

and as before this yields an Egyptian fraction expansion of $\frac{4}{4r+2}$ if and only if the denominators $r + 1$ and $(r + 1) \cdot (4r + 3)$ are distinct. This follows because $4r + 3 \geq 3$.

Continuing as in the previous cases we may write

$$\frac{4}{4r+1} - \frac{1}{r+1} \quad = \quad \frac{3}{(r+1)\,(4r+1)} \ .$$

There are two cases depending upon whether 3 divides the denominator. Suppose it does. Then the equation displayed above will define an Egyptian fraction expansion as a sum of two unit fractions if and only if the right hand side is not equal to $1/r+1$. However, since $4r+1 \geq 5$ we have

$$\frac{1}{r+1} > \frac{3}{5\,(r+1)} \geq \frac{3}{(r+1)\,(4r+1)}$$

and this proves the desired inequality. Now the same inequality also holds regardless of whether 3 divides $(r+1)\,(4r+1)$. In the case where 3 does not divide this number, the fraction

$$\frac{3}{(r+1)\,(4r+1)}$$

has an Egyptian fraction expansion with three distinct terms. Now the fraction itself is less than $1/r+1$, and therefore we see that

$$\frac{4}{4r+1} = \frac{1}{r+1} + \frac{3}{(r+1)\,(4r+1)}$$

is a sum of $1/r+1$ with three other unit fractions such that (i) the three unit fractions are distinct, (ii) each is strictly less than the term $1/r+1$. It follows that the right hand side yields an Egyptian fraction expansion of $4/4r+1$ with four distinct terms.

To conclude the discussion of this case, we need to show that some fractions of the form $\frac{4}{4r+1}$ cannot be written as a sum of two unit fractions with distinct denominators. Suppose that $r=1$ so that the fraction in question is $\frac{4}{5}$. In the spirit of the previous discussion, suppose that we have

$$\frac{4}{5} = \frac{1}{a} + \frac{1}{b}$$

where $a > b$; as before these equations and $b > 0$ imply

$$\frac{2}{a} > \frac{4}{5} > \frac{1}{a}$$

and since a is an integer the only possibility is $a = 2$. The proof that $\frac{4}{5}$ does not have an Egyptian fraction expansion with two terms thus reduces to the following computation:

$$\frac{4}{5} - \frac{1}{2} = \frac{3}{10}$$

Since the term on the right hand side is not a unit fraction, the assertion regarding expansions of $\frac{4}{5}$ follows immediately.■

Note that similar considerations show that $\frac{4}{13}$ does not have an Egyptian fraction expansion with two distinct terms. On the other hand, we have

$$\frac{4}{9} = \frac{1}{3} + \frac{1}{9}\,.$$

Here is a question that should be easy to answer:

What is the smallest integer of the form $4r+1 > 9$ such that $4/4r+1$ has an Egyptian fraction expansion with two distinct terms?

Final remark. The question of Erdős is whether there are any examples for which one actually needs four terms. As indicated in the notes, the answer is not known.