2. Greek mathematics before Euclid

(Burton, **3.1 – 3.4, 10.1**)

It is fairly easy to summarize the overall mathematical legacy that Greek civilization has left us: Their work changed mathematics from a largely empirical collection of techniques into a subject with a coherent organization based upon deductive logic. Perhaps the best known consequence of this transformation is the very strong emphasis on justifying results by means of proofs. One important advantage of proving mathematical statements is that it dramatically increases the reliability and accuracy of the subject and the universality of its conclusions. These features have in turn greatly enhanced the usefulness of mathematics in other areas of knowledge.

Logic and proof in mathematics

Before discussing historical material, it seems worthwhile to spend a little more time discussing the impact of the Greek approach to mathematics as a subject to be studied using the rules of logic. This is particularly important because many often repeated quotations about the nature of mathematics and the role of logic are either confusing or potentially misleading, even to many persons who are quite proficient mathematically.

Inductive versus deductive reasoning. Earlier civilizations appear to have reached conclusions about mathematical rules by observation and experience, a process that is known as *inductive logic*. This process still plays an important role in modern attempts to understand nature, but it has an obvious crucial weakness: A skeptic could always ask if there might be some example for which the alleged rule does not work; in particular, if one is claiming that a certain rule holds in an infinite class of cases, knowledge of its validity in finitely many cases need not yield any information on the remaining infinite number of cases.

Some very convincing examples of this sort are given by a number-theoretic question that goes back nearly two thousand years: If p is a prime, determine whether there are integers m and n such that

$$m^2 = p \cdot n^2 + 1 .$$

This identity is known as Pell's equation; as noted above, this equation had been recognized much earlier by Greek mathematicians and several Hindu mathematicians had studied it extensively during the thousand years before Pell's work in the seventeenth century. Here is a MacTutor reference:

http://www–groups.dcs.st–and.ac.uk/~history/HistTopics/Pell.html

If we take $p = 991$ and compute $991 \cdot n^2 + 1$ for a even a few million values of n, it is easily to conclude that $991 \cdot n^2 + 1$ will never be a perfect square, and in fact this is true until one reaches the case

$$n \;=\; 12\,055\,735\,790\,331\,359\,447\,442\,538\,767 \quad \sim \quad 1.2 \times 10^{29}$$

and for this value of n the expression does yield a perfect square. An even more striking example appears in J. Rotman's undergraduate text on writing mathematical proofs: The smallest n such that

$$1000099 \cdot n^2 + 1$$

is a perfect square has 1115 digits. Here is a bibliographic reference for Rotman's book:

> J. Rotman, ***Journey into mathematics. An introduction to proofs.*** *Prentice Hall, Upper Saddle River, NJ,* 1998. ISBN: 0–13–842360–1.

One could similarly compute the expression $n^3 - n$ for several million values of n and conclude that this expression is probably divisible by 6 in all possible cases. However, in this case one can conclude that the result is always true by the sort of argument we shall now sketch: Given two consecutive integers, we know that one is even and one is odd. Likewise, if we are given three consecutive integers, we can conclude that exactly one of them is divisible by 3. Since

$$n^3 - n \;=\; n \cdot (n + 1) \cdot (n - 1)$$

we see that one of the numbers n and n + 1 must be even, and one of the three numbers n – 1, n and n + 1 must be divisible by 3. These divisibility properties combine to show that the entire product is divisible by $2 \cdot 3 = 6$.

One ***important and potentially confusing point*** is that the deductive method of proof called *mathematical induction* is **NOT** an example of inductive reasoning. Due to the extreme importance of this fact, we shall review the reasons for this: An argument by mathematical induction starts with a sequence of propositions $\mathbf{P}_n$ such that

(1) $\mathbf{P}_1$ is true,
(2) for all positive integers n, if $\mathbf{P}_n$ is true then so is $\mathbf{P}_{n+1}$,

and concludes that every statement $\mathbf{P}_n$ is true. This principle is actually a proof by *reductio ad absurdum*: If some $\mathbf{P}_n$ is false then there is a least n such that $\mathbf{P}_n$ is false. Since $\mathbf{P}_1$ is true, this minimum value of n must be at least 2, and therefore n – 1 is at least 1. Since n is the first value for which $\mathbf{P}_n$ is false, it follows that $\mathbf{P}_{n-1}$ must be true, and therefore by (2) it follows that $\mathbf{P}_n$ is true. So we have shown that the latter is both true and false, which is impossible. What caused this contradiction? The only thing that could be responsible for the logical contradiction is the assumption that some $\mathbf{P}_m$ is false. Therefore there can be no such m, and accordingly each statement $\mathbf{P}_n$ must be true

The accuracy of mathematical results. In an earlier paragraph there was a statement that proofs greatly increase the reliability of mathematics. One frequently sees stronger assertions that proofs ensure the absolute truth of mathematics. It will be useful to examine the reasons behind these differing but closely related viewpoints.

Perhaps the easiest place to begin is with the question, "What is a geometrical point?" Mathematically speaking, it has no length, no width and no height. However, it is clear that no actual, observable object has these properties, for it must have measurable dimensions in order to be observed. The mathematical concept of a point is essentially a theoretical abstraction that turns out to be extremely useful for studying the spatial properties of the world in which we live. This and other considerations suggest the following way of viewing the situation: Just like other sciences, mathematics is formally a theory about some aspects of the world in which we live. Most of these aspects involve physical quantities or objects – concepts that are also fundamental to other natural sciences.

If we think of mathematics as dealing only with its own abstract concepts, then one can argue that it yields universal truths. However, if we think of mathematics as providing information about the actual world of our experience, then a mathematical theory must be viewed as an idealization. As such, it is more accurate to say that the results of mathematics provide extremely reliable information and a degree of precision that is arguably unmatched in other areas of knowledge. The following quotation from Albert Einstein summarizes this viewpoint quite well:

> As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Given the extent to which Einstein used mathematics in his work on theoretical physics, it should be clear that this comment did not represent a disdain for mathematics on his part, and in order to add balance and perspective we shall also include some quotations from Einstein supporting this viewpoint:

> One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts.
>
> But there is another reason for the high repute of mathematics: It is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.

This reflects the high degree of reliability that mathematics possesses due to its rigorous logical structure.

<u>The place of logic in mathematics</u>. Despite the importance of logic and proof in mathematics as we know the subject, it is important to remember that all the formalism is a means to various ends rather than an all-encompassing end in itself. Just like other subjects, mathematical discovery follows the pattern described by I. Kant (1724–1804) in his *Critique of Pure Reason*:

> All our knowledge begins with the senses, proceeds then to the understanding, and ends with reason.

In particular, the discovery in mathematics uses experience and intuition to develop concepts and ideas, and the validity of the latter is determined by means of deductive logic. The following quote from H. Weyl (1885–1955) summarizes this use of logic to confirm the reliability of mathematical conclusions:

> Logic is the hygiene the mathematician practices to keep his ideas healthy and strong.

In everyday life, different standards of hygiene are appropriate for different purposes. Clearly the tight standards of hygiene necessary for manufacturing computer chips are different from the standards that are reasonable for repairing computers. The same general principle applies to logical standards for the study and uses of mathematics.

The role of definitions in formal logic. In logical arguments it is important to be careful and consistent when stating definitions. This contrasts with everyday usage, where it is often convenient to be somewhat imprecise in one way or another. For example, if one looks up a definition in a standard dictionary and then looks up the definitions of the words used to define the original word and so forth, frequently one comes back to the original word itself, and thus from a strictly logical viewpoint the original definition essentially goes around in a circle. Often such rigorous definitions of words in mathematics have implications that are contrary to standard usage; the following quotation from the twentieth century English mathematician J. E. Littlewood (1885–1977) illustrates this phenomenon very clearly:

> A linguist would be shocked to learn that if a set is not closed this does not mean that it is open, or again that "E is dense in E" does not mean the same thing as "E is dense in itself."

The rigidity of mathematical definitions is described very accurately in a well known quotation of C. Dodgson (better known as Lewis Carroll) that appears in his classic book, ***Through the looking glass*** (Alice in Wonderland):

> "When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean – neither more nor less."

There will be further comments on logical definitions later in this unit.

The evolution of logical standards over time. Very few things in this world emerge instantly in a fully developed form and remain completely unchanged with the passage of time. The logical standards for mathematical proofs are no exception to this. Ancient Greek and Roman writings contain some information on the development of logic and mathematical proofs in Greek civilization. Some aspects of this process in Greek mathematics will be discussed later in these notes, and we shall also discuss the changing standards for mathematical proofs at various points in subsequent units. This continuing refinement of logical standards is frequently related to advances within mathematics itself. When mathematicians and others make new discoveries in the subject and check the logical support for these discoveries, occasionally it is apparent that existing criteria for valid proofs require a careful re-examination. Often such work is absolutely necessary to ensure the accuracy of new discoveries. In some cases such refinements of logical standards raise questions about earlier proofs, but in practice mathematicians are able to address such questions effectively with relatively minor adjustments to previous arguments.

During the past thirty years, some uses of computers in mathematical proofs have raised unprecedented concerns. Perhaps the earliest example to generate widespread attention was the original proof of the Four Color Theorem by K. Appel and W. Haken in the middle of the nineteen seventies. The most intuitive formulation of this result is that four colors suffice to color a "good" map on the plane (each country consists of a single connected piece, and no boundary point lies on the boundaries of more than three countries; in particular, this eliminates phenomena like the four corners point in the U.S. where Colorado, Utah, Arizona and New Mexico meet – one can always modify boundaries very slightly to achieve this condition). This original proof used a computer to analyze thousands of examples of specific maps, and questions arose about the reliability of such a program. One widely held view reflects a basic principle of the Scientific Method regarding experimental results: In order to verify them, someone else should be able to reproduce the results independently. For computer assisted proofs, this means running another computer test on a different machine using independently written programs. In fact, tests of this sort were done for the Four Color Theorem with positive results.

Comment on the term "Greek mathematics"

When one discusses any aspect of ancient Greek culture, it is important to remember that the latter became the dominant intellectual framework over an increasing number of geographic areas as time progressed, and particularly in the Hellenistic period – which is conveniently viewed as beginning with the conquests of Alexander the Great and the founding of Alexandria – it includes contributions from many different geographical areas and nationalities in Southern and Eastern Europe, Western Asia, and Northern Africa. The term "Greek mathematics" should be interpreted in this sense.

The beginnings of Greek mathematics

The early period of Greek mathematics began about 600 B. C. E., well over a thousand years after the period during which most surviving documents from Egyptian and Babylonian mathematics were written. However, in contrast to the primary sources we have for these cultures, our information about the earliest Greek mathematics comes from secondary sources. The writings of Proclus (410 – 485 A. D.) are particularly useful about this period and make numerous references to a lost history of mathematics written by a student of Aristotle named Eudemus of Rhodes (350 – 290 B. C. E.) about 325 B. C. E.

What, then, can we say about the beginnings of Greek mathematics? We can conclude that Greek civilization learned a great deal about Egyptian and Babylonian mathematics through direct contacts which included visits to these lands by Greek scholars during the sixth century B. C. E. We can also conclude that during this period the Greeks began to organize the subject using deductive logic – a development that has had an obvious an enormous impact both on mathematics and on other areas of human knowledge. We can also conclude that certain individuals like Thales of Miletus and Pythagoras of Samos played prominent roles in the development of the subject, both through their own achievements and through the schools of study which they led. We can also safely conclude that certain results were known during the sixth century B. C. E. However, we cannot be certain that all the biographical stories about these early mathematicians are accurate, and there is a considerable uncertainty about the proper attribution of results, quotations, and specific achievements to individuals. Our discussions of the earliest Greek mathematicians should be viewed in this light. In particular, it is probably better to view the progress during this period very reliably as the legacy of a culture and less reliably as the legacy of specific individuals who became legendary figures.

Thales of Miletus (*c.* 624 – 548 B.C.E.) is the first individual to be credited with specific mathematical discoveries and contributions. Regardless of whether the attributions of various proofs to him listed on page 83 of Burton are correct, it seems clear that Thales contributed to the organization of mathematical knowledge on logical rather than to empirical grounds. Thales is also credited with using basic ideas about similar triangles to make indirect measurements in situations where direct measurements were difficult or impossible. Two examples, mentioned on pages 83 – 85 of Burton, involve measuring the height of the Great Pyramid by means of shadows and finding the distance from a boat to the shoreline.

We have already mentioned that Babylonian mathematicians were acquainted with the formula we know as the Pythagorean Theorem, and although it seems clear that the Pythagorean school knew the result quite well, there is no firm evidence whether or not they actually found a proof of this result; in fact, the popular story about sacrificing an animal in honor of the discovery is totally inconsistent with Pythagorean philosophy, and as such it has little credibility. However, Pythagoras of Samos (*c.* 580 – 500 B.C.E.) and his school had a major impact on the development of mathematics that we shall now discuss. Given that the Pythagorean school was very reclusive, it is particularly difficult to make any attributions of their work to specific individuals.

One major contribution of the Pythagorean school was their adoption of mathematics as a fundamental area of human knowledge. In fact, classical writings indicate that mathematics was their foundation for an all-encompassing perspective of the world, including politics, religion, and philosophy. Their program of study consisted of number theory, music, geometry and astronomy.

The Pythagoreans were intensely interested in properties of numbers, and many of their speculations on the philosophical properties of numbers indicate a strong tendency towards mysticism. However, this fascination with numbers led to the discovery of many interesting and important relationships, including some that are still sources of unsolved problems.

Although there are questions whether the Pythagorean school actually gave a proof for the result we call the Pythagorean Theorem, it is clear that their studies of this result yielded some important advances. Certainly the most far-reaching was the discovery that the square root of 2 is not rational. Subsequently others recognized addutuibak examples of irrational square roots, and Theaetetus (*c.* 417 – 369 B.C.E.) proved the definitive result: *The square root of* n *is never rational unless* n *is a perfect square*.

The existence of such irrational numbers had an enormous impact on the development of ancient Greek mathematics, and it is largely responsible for the Greek emphasis on geometrical rather than algebraic methods. In particular, Greek mathematics made a clear distinction between "numbers" which were ratios of positive integers and geometrically measurable magnitudes that included quantities like **sqrt** (2). The relation between these two concepts continued to be a source of difficulties for Greek mathematics until much later work by Eudoxus of Cnidus (408 – 355 B.C.E.) that we shall discuss in the next unit on Euclid's *Elements*.

Burton discusses several other aspects of numbers that the Pythagoreans reportedly studied. Two specific contributions involve the concepts of *perfect numbers* and *amicable pairs*. Given two positive integers b and c, we shall say that b evenly divides c if c is an integer multiple of b, and we shall say that b is a proper divisor of c if b is strictly less than c; a positive integer n is said to be perfect number if it is equal to the sum of its proper divisors. The first two perfect numbers are 6 = 1 + 2 + 3 and 28 = 1 + 2 + 4 + 7 + 14. Euclid's Elements contains the following general method for constructing perfect numbers: If 2^{p-1} is prime, then

$$2^{p-1} \cdot (2^p - 1)$$

is a perfect number. A proof of this result appears on pages 472 – 473 of Burton, and, as noted there, a result of L. Euler from the eighteenth century shows that every even perfect number has this form. It is not known whether odd perfect numbers exist; results to date show that there are no such numbers less than 10^{300}.

The description of even perfect numbers leads naturally to the following question: *For which integers* p *is* $2^p - 1$ *a prime number?* Simple algebra shows this can only happen if p is prime, but in 1536 Hudalricus Regius showed that the integer $2^{11} - 1 = 2047$ is equal to $23 \cdot 89$, and in fact there are many primes p for which $2^p - 1$ is not prime. A prime number of this form is called a *Mersenne prime*. During the past 50 years, computer calculations have expanded the list of known Mersenne primes from 17 to 42, with the most recent addition to the list announced and independently verified (by two other separate computations) in February of 2005. The prime number has 7,816,230 digits, and the original proof of this result took more than 50 days of calculations on a 2.4 GHz Pentium 4 computer. This information is taken from the following source:

http://www.mersenne.org/prime.htm

A pair of positive integers is said to be an *amicable pair* if each is equal to the sum of the proper divisors of the other. Although it might not be obvious that such pairs exist, the Pythagoreans reportedly knew that 220 and 284 form an amicable pair. Further material on this topic appears on page 478 of Burton. An extremely current summary of known results (up to March, 2005) appears on the following site:

http://mathworld.wolfram.com/AmicablePair.html

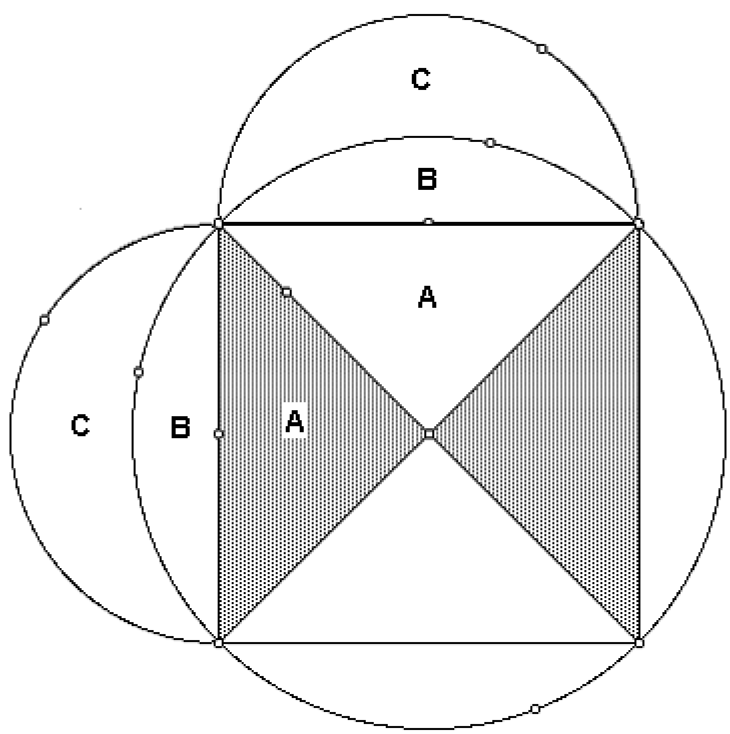
Final note: There is evidence that the amicable pair {17296, 18416} discovered by Fermat in the seventeenth century had been known to Thabit ibn Qurra (826 – 901), who gave a general criterion for recognizing amicable pairs that is stated on page 478 of Burton. As noted on that page, we do not have a characterization of amicable pairs comparable to the simple criterion for even perfect numbers given in the writings of Euclid and Euler.

The "Heroic Age" and its aftermath

This period covers all of the fourth century B. C. E. and most of the following century. The dominant influences during the period were various groups of scholars, particularly the Elean, Sophist and Platonic schools.

Probably the most widely known member of the Elean school was Zeno of Elea (490 – 430 B. C. E.), whose challenging paradoxes about moving objects have continued to attract attention and cause discomfort ever since they were first stated. These problems illustrate the difficulties that arise if one is too casual about mixing discrete and continuous physical models. One of them ("Achilles and the tortoise") is described on page 97 of Burton; it purports to show that the faster Achilles will never overtake the slower tortoise. Clearly the conclusion is absurd, but finding the flaw in the argument requires ideas unknown to the ancient Greeks. All of Zeno's arguments involve a sequence of time intervals such that each is half the preceding one, and in modern language Zeno's paradoxes implicitly assume that the sum of all these terms diverges. Of course, today we know that the sum of the terms is just twice the initial time interval, but the study of convergent infinite series began about 1500 years after the paradoxes were first stated (specifically, during the thirteenth century in China and the fourteenth century in Europe).

The work of Hippocrates of Chios (470 – 410 B. C. E., not to be confused with the physician Hippocrates of Cos) illustrates the evolution of Greek mathematics between the time of the Pythagoreans and the later eras of Plato and Euclid. He took important steps towards a systematic development of geometry from an axiomatic viewpoint, he was one of the first persons credited with using the technique of proof by contradiction (*reductio ad absurdum*), and he wrote the first text on the elements of geometry more than a century before Euclid's *Elements*. Hippocrates is also recognized for his results on computing the areas of certain geometric figures known as *lunes*; these are plane regions bounded by a pair of circular arcs with different radii. In the figure below, the regions marked with C are lunes determined by two circles such that the diameter of one is **sqrt** (2) times the diameter of the other.



Hippocrates' determination of the area C bounded by either lune proceeds as follows (compare pages 117–119 of Burton): He knew that the area of a semicircular region was proportional to the square of the radius. Therefore the area of the larger semicircle, which is 2 (A + B) must be twice that of the smaller semicircle, which is B + C; therefore we have 2 (A + B) = 2 (B + C). Simple algebra now tells us that A = C. Given the irregular shape of the lune, it is not immediately obvious that its area should be given by a simple expression, but the argument shows the lune's area can be expressed very simply. Of course one can use integral calculus to express the area of the lune as a definite integral, and computing the latter is an interesting and somewhat challenging exercise.

Hippocrates' proof provides some interesting insights into the level of Greek geometry at the time. First, it indicates that Greek mathematics had attained a fairly good level of proficiency in manipulating geometric quantities during this period. Second, it illustrates the usefulness of deductive reasoning to discover information that is not intuitively obvious and not likely to be discovered by empirical means. Third, the argument does not really need the explicit computation of the area of the region enclosed by a circle of radius r (area $= \pi r^2$) which probably was not known at the time, but instead it uses just a weaker proportionality statement; as such, the proof indicates an ability to find ways of working around obstacles to reach the objective. However, Hippocrates' result is also noteworthy because of its relation to the three "impossible" construction problems of antiquity, which had already attracted considerable attention in Greek mathematics before Hippocrates' work. Informal statements of the three problems are well known, but for our purposes it is worthwhile to state them a little more precisely.

<u>The three "impossible" classical construction problems.</u> Using only an **unmarked** straightedge (not a ruler!) and a collapsible compass, carry out the following constructions:

1. Trisect an arbitrary angle.
2. Find a square whose area is equal to that of a given circle.
3. Find a cube whose volume is twice that of a given cube.

We shall say more about these later, and an explanation of why these problems are impossible to solve is given in a supplement to these notes; for now we simply note that Hippocrates' results on lunes were byproducts of efforts to solve the second problem.

The Sophists were particularly interested in these three construction problems. Competing schools of thought at the time were strongly critical of the Sophists for several reasons, but for our purposes these controversies will be ignored and we shall concentrate on mathematical achievements. One of the best known Sophists was Hippias of Elis (460 – 400 B. C. E.), who made an early and significant contribution to these problems; specifically, the introduction of curves other than straight lines and circles. This curve, the *quadratrix* or *trisectrix of Hippias*, is discussed from the classical viewpoint on pages 125 – 127 of Burton. Its equation in polar coordinates is

$$r = \theta / \sin \theta,$$

and its equation in Cartesian coordinates is

$$x = y \cdot \cot y.$$

The discussion in Burton indicates how this curve can be used to trisect angles and find a square whose area equals that of a given circle. Further information on this curve may be found at the online sites listed below. The second site has a link to an animated graphic tracing the motion of an object along the curve which corresponds to the classical Greek definition.

http://www-groups.dcs.st-and.ac.uk/~history/Curves/Quadratrix.html

http://xahlee.org/SpecialPlaneCurves_dir/QuadratrixOfHippias_dir/quadratrixOfHippiasGen.mov

Classical writers assert that Hippias used the curve to trisect angles and the application to squaring circles was completed later by Dinostratus (390 – 320 B. C. E.), but others have claimed that this was also known to Hippias. In any case, Greek matheamaticians developed many different curves that could be used to solve the three classical construction problems, and some of these curves have proven to be extremely important in mathematics and its applications to physics. We shall return to this in the unit on Greek mathematics after Euclid.

Finally, we come to the Platonic school. In any discussion of ancient Greek knowledge and thought, it is nearly impossible to avoid mentioning Socrates, Plato and Aristotle. Although Socrates was uninterested, perhaps even negative, about mathematics, Plato and his students had a major impact on the subject. Plato himself was not a mathematician, but his views had a major influence on the subject in several ways.

(1) He insisted on a more rigorous logical framework for doing mathematics, including carefully formulated definitions, axioms, postulates and strict sequential development.
(2) His idea of viewing mathematics formally as an idealized model for reality became a standard for future thought,
(3) His insistence on constructions by straightedge and compass became an ideal for much future work on the subject even though there was also much work on solving the three "impossible" classical construction problems by other methods.
(4) His emphasis on some aspects of solid geometry, particularly on the five regular Platonic solids, spurred further interest in this area.

We shall say more about the five regular Platonic solids later, but for the time being we describe them briefly. The most basic are the *cube* and the *tetrahedron*; the latter is a triangular pyramid whose base and sides are all equilateral triangles. The centers of the six faces of a cube are the vertices of another regular solid called the *octahedron*, which consists of eight equilateral triangles, with each vertex lying on exactly four of them. One also has the *dodecahedron*, which is a configuration of regular pentagons having twelve faces such that each vertex lies on three of the faces, and the *icosahedron*, which consists of twenty equilateral triangles with five meeting at each vertex.

The specific mathematical contributions of the Platonic school are due to some of Plato's students who pursued mathematics as well as some of the latter's students. We have already mentioned some contributions of Theaetetus to the study of irrational numbers, and in the section on Euclid we shall discuss the contributions of Eudoxus of Cnidus. The latter is also known for his approach to computing the area bounded by a circle by using inscribed and circumscribed regular polygons; specifically, the idea was that one obtained increasingly better approximations by taking polygons with greater numbers of sides, and in the limit the areas enclosed by these polygons became the area bounded by the circle. This *method of exhaustion* very clearly anticipated the methods of integral calculus for finding areas using approximations by more manageable figures. Although Aristotle was not a mathematician, his work on logic further refined the role of that subject as a formal setting for mathematics.

Addenda to these notes

There are three separate items. The first (**2A**) describes a method of computing the area of Hippocrate's lune using integral calculus, the second (**2B**) discusses the reasons why the three classical construction problems cannot be solved by means of an unmarked straightedge and a compass, and the third (**2C**) discusses one incorrect attempt to trisect angles. Further information on the latter can also be found at the following site:

http://www.jimloy.com/geometry/trisect.htm

In particular, the portion called

Part III - Close, But No Banana

specifically discusses further elaborations of the construction in (**2C**). The entire Jim Loy mathematics site

http://www.jimloy.com/math/math.htm

also treats many other topics that are relevant to high school and lower level college mathematics in an accessible and mathematically correct manner, and it is highly recommended.