2.A. Calculus and the area of Hippocrates' lune

The methods of integral calculus provide techniques for computing the areas of arbitrary lunes, so we shall indicate how this applies to the special example computed by Hippocrates of Chios. We shall refer to the figure in the main unit.

We shall take the smaller semicircular region to be the one bounded by the circle $x^2 + y^2 = 2$ and the x-axis. Having made that choice, we must take the second circle to be the one whose radius is $\sqrt{2}$ times the given circle and contains the end points of the semicircle. Since the radius of the original circle is $\sqrt{2}$ and the end points are $(\pm\sqrt{2}, 0)$, we have enough information to write down the equation of the circle, and in fact it is given as follows:

$$x^2 \;+\; (y + \sqrt{2})^2 \quad = \quad 4$$

The smaller circle is the upper curve for this region and the larger circle is the lower curve. The equations for the pieces of these curves in the upper half plane are given by $y = \sqrt{2 - x^2}$ and $y = \sqrt{4 - x^2} - \sqrt{2}$. The region of interest to us lies between the vertical lines $x = \pm\sqrt{2}$. Therefore the area of the lune is given by the following integral:

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{2 - x^2} + \sqrt{2} - \sqrt{4 - x^2} \right) \, dx$$

It might be worthwhile to compute this definite integral and see how easy or difficult it is to retrieve Hippocrates' result using calculus. Of course, one could also apply this to other lunes that Hippocrates studied as well.