3. Euclid and the *Elements*

(Burton, **4.1 – 4.3**)

Alexander the Great's political empire fragmented shortly after his death in 323 B. C. E., but the cultural effects of his conquests were irreversible and defined the course of future civilization. Greek culture became an established framework for many areas of knowledge in the Mediterranean world, and mathematics was a particularly important example. The founding of Alexandria in Egypt during and immediately after Alexander's time reflects this change very clearly, for the city became a center of learning for the entire Hellenistic culture.

Euclid of Alexandria (325 **–** 265 B. C. E.) was one of the first mathematicians based in that city. Of course, he is mainly known for organizing and presenting the basics of mathematical knowledge at the time into the thirteen volume work called the *Elements*, and because of this monumental work he is arguably the best known of all the ancient Greek mathematicians.

The achievements and influence of the ***Elements***

Few books ever written have circulated as widely around the world as Euclid's *Elements*, and much has been written about the advances in knowledge it represents and its continued significance ever since it was written. Not surprisingly, some discussions of the *Elements* are more accurate and objective than others. Therefore we shall begin by describing the features of the *Elements* that are generally regarded as the reasons it has remained such an important work for such a long period of time.

As noted before, the purpose of the Elements was to give a systematic account of many basic and major mathematical results that were known at the time. The work was not meant to be a complete account of mathematical knowledge at the time; in the words of Proclus, the work covers "those theorems whose understanding leads to knowledge of the rest."

Most of the material in the *Elements* had been well known to the Greeks before Euclid's time. However, the overall organization of the *Elements* was due to Euclid, and he personally created many of the proofs that appear in this work. In some cases these arguments and formulations probably filled gaps or weak spots in theorems and proofs that were known at the time.

The single most important aspect of the Elements is its logical organization, which begins with definition for important concepts, formulates some basic properties of these concepts that will be assumed, and uses deductive logic to prove new conclusions or theorems. These results were presented in a formally logical order, with each proof depending only on the results and assumptions that had appeared previously in the work. Even if a statement appeared to be completely obvious, the rigorous logical structure demanded a formal proof.

This axiomatic method of approaching a subject is the standard for scientific reasoning that is still used today, and it has also been used in a wide range of other disciplines over the ages. In particular, many philosophical writings over the centuries have used the *Elements* as model for their reasoning; some particularly notable examples include Thomas Aquinas and numerous rational philosophers especially the seventeenth and eighteenth centuries (*e.g.*, very explicitly in the writings of B. Spinoza). Given the influence of the *Elements* on future work in many areas of human knowledge, it is not surprising that some view it as the most successful textbook ever written.

Euclid's formulations of various mathematical results quickly superseded some earlier ones and became established as definitive in many cases; with the passage of over 2300 years, some of his versions have in turn been superseded by others for various reasons, but there are also many instances where his formulations and arguments are still the preferred ones today. This is particularly true in elementary geometry, but it is also true in many parts of elementary number theory. In each subject, quite a few of Euclid's methods and ideas are still present to some extent, not for any sentimental value they may have but simply because they are still the most clear and direct ways to consider certain topics.

Euclid's logical framework for geometry is very concise and powerful, and his work was long believed many to provide an absolutely true description of the physical world and everything that one needs to understand it. Questions along these lines attracted a great deal of attention from philosophers, especially during the seventeenth and eighteenth centuries, but several distinct mathematical developments during the nineteenth and twentieth centuries have shown that the geometric structure of the physical world is too complex to be described entirely and precisely in Euclidean terms. However, small pieces of the physical world often appear to obey the rules of Euclidean Geometry up to a high degree of accuracy.

Our knowledge of Euclid himself is extremely limited, and most of it comes from the writings of Proclus. There are some often repeated anecdotes about him that appear in the paragraph beginning at the bottom of page 137 in Burton.

Subjects covered in the ***Elements***

We have already mentioned that there are thirteen parts to Euclid's *Elements* (formally called books, but they really are more like individual chapters of a single book). A more detailed table of contents for the *Elements* is available online at the site

http://aleph0.clarku.edu/~djoyce/java/elements/toc.html

(in fact, the latter contains links to the entire *Elements*). We shall summarize some of the main topics covered in this work.

The first few books treat basic plane geometry, and the material in these books has served as the core content of geometry courses for most of the past 2300 years. In particular, the first two books cover the basic properties of rectangles and triangles. A great deal of the second book is devoted to geometric proofs of results that we now view as essentially algebraic; this reflects the philosophical problems that ancient Greek mathematicians had with irrationals. It appears that much of the material in Book I was originally developed by Thales and the Pythagoreans, although in some cases the proofs seem clearly due to Euclid. There is a lengthy commentary on the contents of the first two books on pages 139 – 161 of Burton; we shall say more about several aspects of this material later. Pages 157 – 160 are devoted to the so-called ***golden section problem*** that has preoccupied mathematicians and others since the time of the ancient Greeks. In geometric terms, given a segment **AB**, the objective is to find a point **C** between **A** and **B** such that the lengths of segments satisfy $|\mathbf{AC}| > |\mathbf{CB}|$ and the following proportionality relation:

$$|\mathbf{AC}| \,/\, |\mathbf{AB}| \quad = \quad |\mathbf{BC}| \,/\, |\mathbf{AC}|$$

Algebraically this translates into a quadratic equation for $x = |\mathbf{AC}| / |\mathbf{AB}|$, specifically

$$x^2 + x - 1 = 0$$

and if one solves this equation for the positive root the answer is the number

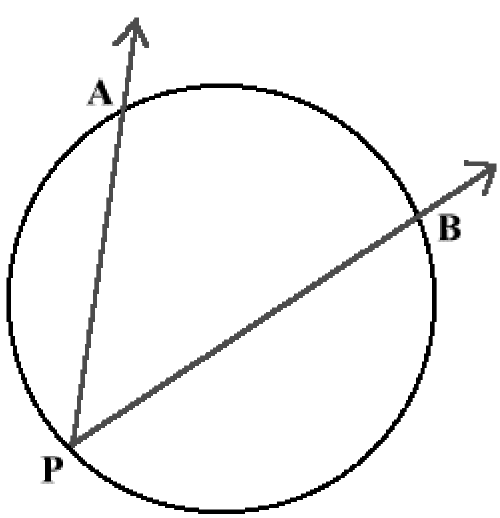
$$x = \tfrac{1}{2}\,(\,\mathrm{sqrt}\,(5) - 1\,) = 0.6180339887498948482045868343656\ldots$$

which is one less than the number • (phi) that plays a central role in a recent popular novel (D. Brown, ***The Da Vinci Code,*** Doubleday, 2003, ISBN 0–385–50420–9). The discussion of Burton also explains the relevance of this number to constructing a regular pentagon by unmarked straightedge and compass.

The third and fourth books study circles and their relations to other geometric figures. Much of this material was apparently known to Hippocrates of Chios. The following result illustrates the types of questions studied in Book III:

The measure of an inscribed angle is equal to one-half the degree measure of its intercepted arc.

In the illustration below, the intercepted arc is the one inside angle $\angle$**ABC**.

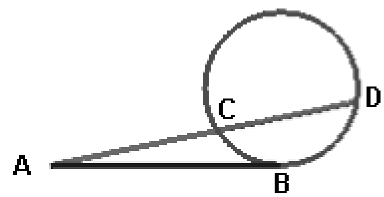


(***Source :*** http://jwilson.coe.uga.edu/emt725/ReviewCir/ReviewCir.htm)

The final result of Book III illustrates how far such questions are pursued:

Suppose that we are given a circle, a point **A** *outside the circle and two lines through* **A** *such that one is tangent to the circle at* **B** *and the other meets the circle at two points* **C** *and* **D** *such that* **C** *is between* **A** *and* **D**. *Then the lengths of the segments satisfy the following equation:*

$$|\mathbf{AB}|^2 = |\mathbf{AC}| \cdot |\mathbf{AD}|$$



Book IV looks more closely at questions like finding circles passing through all three vertices of a triangle (circumscribing a circle about a triangle), finding circles tangent to all three sides of a triangle (inscribing a circle in a triangle), and constructions for certain geometrical figures by unmarked straightedge and compass. The book ends with the construction of a regular 15-sided polygon using only these tools.

Book V covers material that was relatively new at the time. We had already mentioned that the irrationality of **sqrt** (2) was troublesome for Greek mathematicians, and an effective way of working with such numbers was not found until Eudoxus of Cnidus developed an approach to analyzing incommensurable proportions by means of rational numbers. One can rephrase his approach by saying that a geometrical magnitude x is characterized by the following two pieces of rational data:

1. The set of all positive rational numbers strictly less than x.
2. The set of all positive rational numbers strictly greater than x.

We shall discuss this and other aspects of the Condition of Eudoxus in an addendum to these notes. For our present purposes it will be enough to state his criterion for concluding that two ratios of geometrical magnitudes are equal:

The condition of Eudoxus. *Two ratios of (positive real) numbers* a/b *and* c/d *are equal if and only if for each pair of positive integers* m *and* n *we have the following:*

$$ma < nb \quad \textit{implies} \quad mc < nd \quad \textit{and} \quad ma > nb \quad \textit{implies} \quad mc > nd$$

It is worthwhile to stop for a minute and consider what these mean. The first statement is equivalent to saying that for every pair of positive integers m and n we have

$$a/b < n/m \quad \textit{implies} \quad c/d < n/m$$

while the second is equivalent to saying that for every pair of positive integers m and n we have

$$a/b \ > \ n/m \quad \textit{implies} \quad c/d \ > \ n/m\ .$$

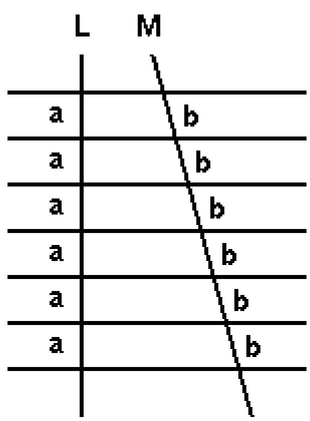
The first of these amounts to saying that if r is a rational number greater than a/b then r is also greater than c/d. As noted in addendum **3.A** to these notes, the first condition implies that a/b $\geq$ c/d. On the other hand, the second amounts to saying that if r is a rational number less than a/b then r is also less than c/d. As noted in the addendum, this implies that a/b $\leq$ c/d. If we put these together we conclude that a/b must be equal to c/d.

In Book VI the Eudoxus theory of proportions for incommensurables is applied to geometrical questions like the classical similarity theorems for triangles. This was a major advance at the time, for before the discoveries of Eudoxus the Greeks were only able to prove similarity theorems for triangles in the ***commensurable*** case: For a pair of triangles **Δ ABC** and **Δ DEF**, this means that the common ratio of the lengths of the corresponding sides

$$|\mathbf{AB}| \,/\, |\mathbf{DE}| \quad = \quad |\mathbf{AB}| \,/\, |\mathbf{DE}| \quad = \quad |\mathbf{AB}| \,/\, |\mathbf{DE}|$$

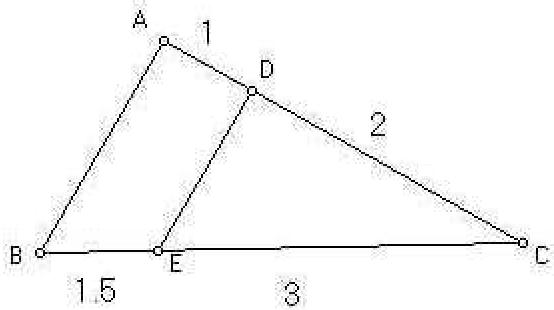
is a ***rational number.*** Greek geometers were able to attack the similarity problem effectively using the following fact, which one might call the ***Notebook Paper Theorem.***

Suppose that we are give a family of distinct parallel lines $\mathbf{P_1}$, $\mathbf{P_2}$, **…** and *two transversals* **L** *and* **M** *that are not parallel or identical to these lines. For each* k *let* $\mathbf{A_k}$ *be the point at which* **L** *and* $\mathbf{P_k}$ *intersect, and let* $\mathbf{B_k}$ *be the point at which* **M** *and* $\mathbf{P_k}$ *intersect. If all the segments* $|\mathbf{A_k A_{k+1}}|$ *have length* a, *for some fixed* a, *then all the segments* $|\mathbf{B_k B_{k+1}}|$ *have length* b *for some fixed* b**.**



Here is an illustration of how one can use the Notebook Paper Theorem in a simple case of similar triangles. The picture below is taken from the following site:

http://www.mccallie.org/myates/chapter_8.htm



Let's suppose we know that the lines **AB** and **DE** are parallel, so that the corresponding angles of **Δ ABC** and **Δ DEC** have equal measurements, and let's suppose also that the lengths of the segments **AC**, **DC** and **EC** are as given in the picture. The basic similarity theorem predicts that the length of the fourth segment **BC** is 4.5 as suggested in the picture, but how do we reach this conclusion? One way to do so is to construct a line through the midpoint **X** of the segment **DC** that is parallel to **AB** and **DE**. This line must meet side **BC** at some point **Y.** We then know that the lengths |**CX**|, |**XD**| and |**DA**| are all equal to 1, and therefore the Notebook Paper Theorem implies that the lengths |**CY**|, |**YE**| and |**EB**| are all equal. Since the length |**EC**| is equal to three, it follows that this common length for the three segments on **BC** must be 1.5, and therefore the length of **BC** must be equal to 4.5. More generally, suppose now that the lengths of **AC** and **DC** were m and n respectively for suitable positive integers m and n. In this case one would construct a whole family of n mutually parallel lines containing **AB** and **DE** such that the segments they cut off on **AC** all have length equal to 1. If the lengths of **EC** and **BC** are equal to p and q respectively, then the same sort of reasoning will show that $q = p \cdot (m / n)$, or equivalently $q / p = m / n$.

Of course this argument breaks down completely if the lengths of **AC** and **DC** are **sqrt(**2**)** and 1, but in these cases it is still possible to prove that |**BC**| = **sqrt (**2**)** · |**EC**| by using the Condition of Eudoxus. Further discussion of such a proof appears in addendum **3.B** to these notes.

Books VII through IX of the *Elements* shift to an entirely different subject; namely, number theory. Here is a partial list of topics:

1. Prime numbers
2. Factorizations of positive integers as products of primes
3. Finding greatest common divisors of numbers using long division of integers (a = bq + r, where r lies between 0 and q – 1)
4. Geometric progressions and their sums
5. The existence of infinitely many prime numbers
6. Perfect numbers, including Euclid's construction of even perfect numbers

A fairly detailed description of key points in these books is given on pages 163 – 174 of Burton.

Book X is an extensive and well organized discussion of irrational numbers, and it is almost certain that it went well beyond previous efforts to understand irrationals. The first proposition in this book is in fact a simple but important observation about limits. In modern language, it says that if we are given a sequence of positive numbers a_n such that $a_{n+1} < a_n / 2$, then the limit of the sequence a_n is equal to zero. This result plays a key role in Eudoxus' derivation of the formula $A = \pi r^2$ for the area of a region bounded by a circle whose radius is equal to r (this result is proved in Book XII).

The final three books (XI – XIII) deal mainly with solid geometry. Book XI extends basic notions of plane geometry to three dimensions; *e.g.*, perpendicularity and parallelism for lines and planes in space, and the various types of solid angles (dihedral, trihedral, etc,) formed by planes. Book XII goes in a quite different direction, using methods of Eudoxus to find areas and volumes of solid figures like cones, pyramids, cylinders and spheres; this book also contains a proof of the familiar area formula $A = \pi r^2$. Finally, Book XIII describes the five regular Platonic solids. The exposition in this final book is less rigorous than in the preceding ones, and some have suggested that it was added afterwards by Euclid or perhaps even someone else. Manuscripts claiming to be later books of the *Elements* exist, but there is a general agreement that these were not written by Euclid.

*Limitations of the **Elements***

Despite the important achievements of the *Elements*, this work is not perfect. While a study of its defects is appropriate, one should also remember that without the model for rigorous logical investigation that the *Elements* provided, it is questionable whether such weak points would have ever been discovered. It is particularly noteworthy that, with a few exceptions, the logical bugs in the *Elements* were not discovered until the nineteenth century.

Man is doomed to err so long as he is striving. – Goethe, *Faust*

Some mathematical historians have criticized the *Elements* for lack of motivation for the subject matter and a similar lack of analyses of the proofs. Such issues can be debated at length (certainly its usefulness as an effective textbook over many centuries seems beyond question), but we shall concentrate on strictly mathematical issues here.

Although the *Elements* devotes considerable effort to formulating careful definitions, at the very beginning it attempts to define too much and later it does not give precise definitions of some basic ideas that are used throughout the work. In mathematics it is not possible to define everything in terms of objects; the chain of definitions must stop somewhere and certain concepts must be taken as primitive or undefined. For example, this applies to the notions of point, lines and planes in most axiom systems for Euclidean geometry; even in cases where one can define some of these, say lines and planes, in terms of other concepts, the latter are generally undefined. A reader usually has (and indeed should have) some intuition about what these concepts represent physically, but none of this can appear in the formal mathematical setting. Further comments about definitions in the *Elements* appear in the third full paragraph on page 139 of Burton and the second full paragraph on page 141.

Euclid's work frequently discusses concepts like two points lying on the same or opposite sides of a line, one point on a line being between two others on the same line, a point lying in the interior of an angle or triangle, and so forth. None of these notions is described formally in the *Elements,* and the corresponding lack of postulates about the properties of these concepts leaves some of the arguments incomplete. We have already mentioned two results from Book III whose proper formulation requires a notion of betweenness for points and a precise description of the interiors of angles. Burton mentions another instance where interiors are needed in Proposition 16 from Book I (the Exterior Angle Theorem; see pages 145 – 146). It is possible to make such ideas logically rigorous by adding additional definitions and assumptions; we shall discuss this later.

Insisting on filling the logical gaps might initially seem needlessly pedantic. No one disputes the basic correctness of the geometric propositions we are considering, so one might ask how much it matters that concepts like betweenness, angle interiors, and points lying on the same or opposite sides of a line were not discussed rigorously. Perhaps this is a no harm, no foul situation. One simple response is that insufficient attention to such concepts can lead to arguments that look very much like those in the *Elements* but have ridiculously false conclusions. On of the best known fallacies is a standard and apparently reasonable "proof" due to W. W. Rouse Ball, which claims to show that every triangle is isosceles. If accepted standards of reasoning lead to such an absurd conclusion, the reliability of any conclusions obtained by such methods must be viewed with suspicion. Here is an excellent online reference for the purported proof that every triangle is isosceles and an analysis of its weak points; the same site also contains a few other geometric fallacies of a similar nature.

http://www.jimloy.com/geometry/every.htm

As noted in the latter, finding the mistake in the argument requires a patient examination of every assertion in the proof related to betweenness and the other notions mentioned above, and in fact the mistake results from using an incorrectly drawn picture to reach false conclusions.

A second source of difficulties in the *Elements* is its use of the so-called "principle of superposition" which says that one can move objects without changing their shape. Apparently Euclid himself was uncomfortable with the idea of proof by superposition, which was used to prove the Side–Angle–Side congruence theorem for triangles. This method was only used at one other point (the proof of the Angle–Side–Angle theorem) even though it could have been used equally well in other instances. The logical framework for the *Elements* says nothing about moving figures around in the plane. Many, perhaps most, axiom systems for Euclidean geometry avoid this problem by assuming the Side–Angle–Side congruence theorem; in cases where this is not done, some other assumption is needed in order to prove this result. The corresponding discussion in Burton begins with the last paragraph on page 144.

Surprisingly, the first logical difficulty in the *Elements* appears in the proof of the very first result on constructing an equilateral triangle with a given base. The basic idea is simple: Starting with a segment **AB**, one constructs a circle with center **A** and radius **AB**, and then one also constructs a circle with center **B** and radius **AB**. Take a point **C** where these circles meet, and this will be the third vertex of an equilateral triangle which has **AB** as one of its sides.

The problem with this argument is that none of the basic assumptions at the beginning of the *Elements* says anything about intersecting circles. In order to complete the proof of the very first proposition, one needs a result saying that if one circle contains one point inside another circle and one point outside that circle, then the two circles have a point in common. A complete mathematical analysis of the situation is a bit too complicated to describe here, so we shall simply note that it is impossible to prove what is needed on the basis of what was assumed. The first two paragraphs on page 142 of Burton discuss this further.

In view of these, it is rather ironic that Euclid had previously been criticized for something that was **not** a mistake; namely, the relatively complicated Fifth Postulate:

> 5. That, if a straight line falling on two straight lines make the interior angle on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

It is particularly significant that this assumption is not needed or used until Proposition 27 of Book I and that a substantial amount of material was developed up to that point. The assumption was avoided in the logical progression until it was no longer possible to do so. For centuries mathematicians and others felt that this fifth postulate, which was far more complicated than any of the others, was a problem that had to be corrected. They felt that it was surely not necessary to include such a lengthy assumption when the others were so short and crisp. Subsequent studies found equivalent assumptions that could be stated more simply (*e.g.*, J. Playfair's seventeenth century version: *Given a line and a point not on it, there is a unique parallel line to the original line passing through the given point*), but none of these appeared to be a consequence of the other assumptions despite many deep and innovative investigations. When several mathematicians in the nineteenth century discovered that the fifth postulate could not be derived as a logical consequence of the other assumptions, or indeed of any version that is modified to overcome the logical inadequacies mentioned above, it became clear that the need for this postulate was an extremely important insight on Euclid's part. One important work leading to the discovery of the fifth postulate's independence was due to G. Saccheri (1677 – 1733), the title of which claims this work vindicates Euclid by proving the fifth postulate from the others. Despite some mistakes at the end, this work was an important contribution, but even so the claim of vindication is somewhat ironic. The real vindication for Euclid is that the fifth postulate is logically indispensable rather than logically redundant.

Finally, it seems worthwhile to mention a point that is not really a logical mistake but apparently a logical insight that was missed. Proposition 5 of Book I proves that if we are given an isosceles triangle Δ **ABC** in which **|AB| = |AC|**, then the measures of the angles **ABC** and **ACB** are equal. Euclid's proof is described on pages 143 – 144 of Burton, and it is a fairly lengthy argument which requires the construction of auxiliary points and line segments. This argument contrasts very sharply with the very short and simple proof discovered by Pappus of Alexandria about 600 years later (see page 145 of Burton). Euclid's argument involves the application of congruence theorems to two separate triangles, and Pappus' argument involves the application of a congruence theorem to a single triangle with the vertices corresponding in a specific "nonstandard" way: **A** corresponds to itself, **B** corresponds to **C**, and **C** corresponds to **B**.

Repairing the logical deficiencies

Given the importance of the *Elements*, it is not surprising that many nineteenth century mathematicians worked to bring the material up to modern logical standards. The most widely known approach was given by D. Hilbert (1862 – 1943) at the end of the nineteenth century. It requires six undefined or primitive concepts (points, lines, planes, betweenness, congruence of segments, congruence of angles) and 21 separate postulates. Here is an online reference:

http://en.wikipedia.org/wiki/Hilbert's_axioms

This axiom system reflects Euclid's approach, which involves minimal use of number systems. However, if one is willing to accept the real number system as given, then it is possible to reformulate the axioms in terms of points, lines, planes, distance between points, and angular measure with fewer assumptions, and with this system of axioms many proofs become much simpler. The basic ideas for this were formulated by G. D. Birkhoff (1884 – 1944) during the second quarter of the twentieth century, and Birkhoff's specific axioms may be found in the link Euclid's Mathematical System at the following online site:

http://www.math.uncc.edu/~droyster/math3181/notes/hyprgeom/hyprgeom.html

The link above also has a listing of Hilbert's axioms.

Other works by Euclid

A few other writings of Euclid have also survived, some in fairly complete forms and other only in fragments. The work most closely related to the *Elements* is called *Data*, which includes numerous further results and problems related to the material in the *Elements* and may be viewed as a sort of supplement to the latter. Other books treating geometrical topics include *On Divisions*, which describes ways of splitting geometrical figures into pieces with prescribed areas, and *Phaenomena*, which discusses spherical geometry as it relates to observations in astronomy. A book on optics has also survived. Several other books have been lost over time; various scholars have attempted to reconstruct portions of these lost works, but views on the accuracy of the attempted reconstructions are mixed.

Further information on these other works, and the *Elements* itself, may be found at the following online site:

http://www.obkb.com/dcljr/euclid.html