3.A. The condition of Eudoxus

Although the irrationality of some numbers like $\sqrt{2}$ was known to the ancient Greeks, they also knew from experience that all geometric magnitudes in their studies could be approximated to any desired degree of accuracy by fractions (*e.g.*, see the rational approximations to $\sqrt{2}$ in Exercise 7 on page 12 of Burton). The approach to studying incommensurable proportions developed by Eudoxus took this one very important step further; its main idea was that a ratio of geometric magnitudes is in fact specifiable in terms of rational numbers that approximate it in an appropriate manner. This idea allowed Greek mathematicians to develop a theory of geometric proportions for incommensurable magnitudes, and it anticipated the rigorous mathematical approach to placing the real number system on a firm logical foundation that was completed by R. Dedekind (1831–1916) in the nineteenth century.

The Condition of Eudoxus is based upon the following crucial property of real numbers:

DENSITY OF RATIONAL NUMBERS. *If x and y are real numbers such that $0 < x < y$, then there is a rational number r such that $x < r < y$.*

Rather than attempt to prove this completely within the formal setting for real numbers, we shall explain why it is true using the standard description of real numbers by means of finite or infinite decimal expansions. If, say, we have $0 < x < y < 1$, then there is some first decimal place, say p, where the p^{th} term x_p in the decimal expansion for x is less than the corresponding term y_p in the decimal expansion for y. If, say, $y_p - x_p > 1$ then one can write down a finite decimal fraction between x and y without much trouble. On the other hand, if the difference between the p^{th} terms is 1 then this becomes more difficult, and rather than attempt this we shall take an alternate approach which is still based upon decimal and base 10 expansions.

The crucial information provided by base 10 expansions is captured by the following result:

ARCHIMEDEAN PROPERTY. *If $x > 0$ then there is a positive integer N such that $N > x$.*

Despite the name for this fact, it was in the *Elements* and Archimedes acknowledged that the result was known to Euclid.

Explanation. The usual base 10 and decimal expansions for a real number have the form $k + m$ where k is a nonnegative integer and $0 \leq k < 1$. If suffices to take $N = k + 1$.■

CONSEQUENCES. (*i*) *If a and b are real numbers such that $0 < a < b$, then there is a positive integer N such that $N \cdot a > b$, and in fact there is a least such positive integer,*

(*ii*) *If h is a positive real number then there is a positive integer M such that $\frac{1}{M} < h$.*

Proofs. For the first statement, by the Archimedean Property we know there is a positive integer N such that $N > b/a$, and if we multiply both sides of this inequality by a we obtain $N \cdot a > b$. But every nonempty set X of positive integers has a least element, so in particular the set of all N satisfying $N\,a > b$ must have a least element.

For the second statement, apply the Archimedean Property to the positive number $1/h$; then we have $1/h < M$. If we take reciprocals of both sides we obtain the inequality $1/M < h$.■

Proof that the rationals are dense in the reals. Let x and y be real numbers such that $0 < x < y$, and choose a positive integer P such that $1/P < y - x$. Next choose the least positive integer Q such that $Q \cdot (1/P) > x$. Since Q is the least such integer we have the inequalities

$$\frac{Q-1}{P} \quad \leq \quad x \quad < \quad \frac{Q}{P} \quad = \quad \frac{Q-1}{P} \quad + \quad \frac{1}{P} \quad < \quad x \;+\; (y-x) \quad = \quad y$$

which proves the existence of a rational number strictly between x and y.■

We now apply this to establish the equality criterion for ratios of positive real numbers.

THE CONDITION OF EUDOXUS. *Let a, b, c, d be positive real numbers. Then*

$$\frac{a}{b} = \frac{c}{d}$$

if and only if the following two statements hold:

(1) *If m and n are positive integers such that $m\,a < n\,b$, then $m\,c < n\,d$.*

(2) *If m and n are positive integers such that $m\,a > n\,b$, then $m\,c > n\,d$.*

Proof. Suppose first that the ratios are equal. Consider the statement (1). If we divide the two inequalities by $m\,b$ and $m\,d$ respectively, we obtain the relations

$$\frac{a}{b} < \frac{n}{m} \qquad\qquad \frac{c}{d} < \frac{n}{m}$$

and (1) holds because $(a/b) = (c/d)$ and the first inequality imply the second. Statement (2) follows by reversing the directions of all inequalities in the argument proving (1).

Now suppose that (1) and (2) hold. If we divide the inequalities as in the previous paragraph by $m\,b$ or $m\,d$ depending upon which variables are present, we see that the following statements are true for all a, b, c, d, m, n:

$$\begin{aligned}
\frac{a}{b} < \frac{n}{m} \quad &\Longleftrightarrow \quad m\,a < n\,b \\
\frac{c}{d} < \frac{n}{m} \quad &\Longleftrightarrow \quad m\,c < n\,d \\
\frac{a}{b} > \frac{n}{m} \quad &\Longleftrightarrow \quad m\,a > n\,b \\
\frac{c}{d} > \frac{n}{m} \quad &\Longleftrightarrow \quad m\,c > n\,d
\end{aligned}$$

Using these equivalences we see that (1) and (2) are respectively equivalent to the following statements:

($1'$) If m/n is a positive rational number such that $(a/b) < (m/n)$, then $(c/d) < (m/n)$.

($2'$) If m/n is a positive rational number such that $(a/b) > (m/n)$, then $(c/d) > (m/n)$.

We may further restate these as follows:

($1''$) The set of all positive rational numbers greater than (a/b) is equal to the set of all positive rational numbers greater than (c/d).

($2''$) The set of all positive rational numbers less than (a/b) is equal to the set of all positive rational numbers less than (c/d).

If a/b were strictly less than c/d, then there would be a positive rational number r such that

$$\frac{a}{b} < r < \frac{c}{d}$$

and this would imply that ($2''$) would be false. Likewise, if a/b were strictly greater than c/d, then there would be a positive rational number s such that

$$\frac{c}{d} < s < \frac{a}{b}$$

and this would imply that $(1'')$ would be false. Since these statements are equivalent to (2) and (1) respectively, it follows that a/b is neither strictly less than c/d nor strictly greater than it. The only remaining possibility is that $(a/b) = (c/d)$, and therefore this statement must be true.■

The Condition of Eudoxus and the Dedekind Completeness Property. For several centuries mathematicians and others have viewed real numbers computationally as objects given by infinite decimal expansions. Results on infinite series from the seventeenth century provided strong evidence that this was a legitimate approach. However, there are a few important questions that arise immediately:

[1] If we choose another base besides 10 for expressing quantities (*e.g.*, the base 2 in which computers ultimately do their calculations or the base 60 of the Babylonians), can we be confident that we have obtained an equivalent system.

[2] Even if we assume that real and rational arithmetic satisfy the usual sorts of algebraic identities like $ab + ac = a(b + c)$ and $\{\ a < b \ \text{ and } \ c > 0 \implies ca < cb\ \}$, how can we be sure that these hold for numbers given by infinite decimal expansions?

Both questions suggest that one should think of real numbers in some conceptual terms besides infinite decimal expansions. In principle, the idea behind Dedekind's approach to the real numbers is that they should be the largest possible algebraic system such that

(*i*) it satisfies the usual rules for arithmetic equations and inequalities,

(*ii*) the rational numbers are dense in it (between any two real numbers one can find a rational number),

(*iii*) it is the largest possible system for which the preceding two properties hold.

The first two of these properties reflect the Condition of Eudoxus, and from this perspective Dedekind's advance was to include everything consistent with these principles.

Of course, any abstract description of the real numbers should yield our everyday description of them by infinite decimal expansions. In fact, it is possible (and relatively straightforward) to prove that real numbers have the usual sorts of infinite decimal expansions and that these behave as expected. A very terse but mathematically complete description of the relation between real numbers and decimals is given in item 1.22 on page 11 of W. Rudin, *Principles of Mathematical Analysis, Third Edition*, McGraw-Hill, 1976. Similar considerations yield analogous expansions if one chooses an arbitrary positive integer ≥ 2 as a computational base rather than 10.

Final notes. A discussion of the difficulties that arise in attempting to answer question [2] directly using infinite decimal expansions can be found in J. F. Ritt, *Theory of Functions*, Kings Crown Press, 1947. Difficulties begin with equations like

$$0.199999999999\ldots \quad = \quad 0.200000000000$$

between different decimal expansions. The problem of describing the reciprocal of an arbitrary positive decimal

$$0.a_1a_2a_3a_4\ldots$$

explicitly in decimal form illustrates the sorts of problems that arise if one tries to use decimals in order to verify that every nonzero real number has a reciprocal.