3.B. Geometric proportions and the Condition of Eudoxus

One important motivation for the Condition of Eudoxus was to consider geometric ratios involving ***incommensurable quantities***; in modern language, these are lengths **|WX|** and **|YZ|** such that the quotient **|WX|** / **|YZ|** is ***not*** a rational number. A basic problem of this nature is depicted in the figure below: In this picture the lines **BD** and **CE** are assumed to be parallel, and one wants to prove that

$$|AB| / |AC| \quad = \quad |AD| / |AE|.$$

If the left hand side is a rational number p/q, then standard manipulations of ratios show that

$$|AB| / p \quad = \quad |AC| / q$$

and ideas discussed in the main notes for this unit then imply that

$$|AD| / p \quad = \quad |AE| / q$$

which then quickly yields **|AD|** / **|AE|** = **p** / **q** = **|AB|** / **|AC|**. Of course this breaks down completely if the ratio **|AB|** / **|AC|** is an irrational number like **sqrt(**2**)**, and we need the Condition of Eudoxus to handle such cases.

Here is a formal statement of Eudoxus' criterion for two ratios to be equal:

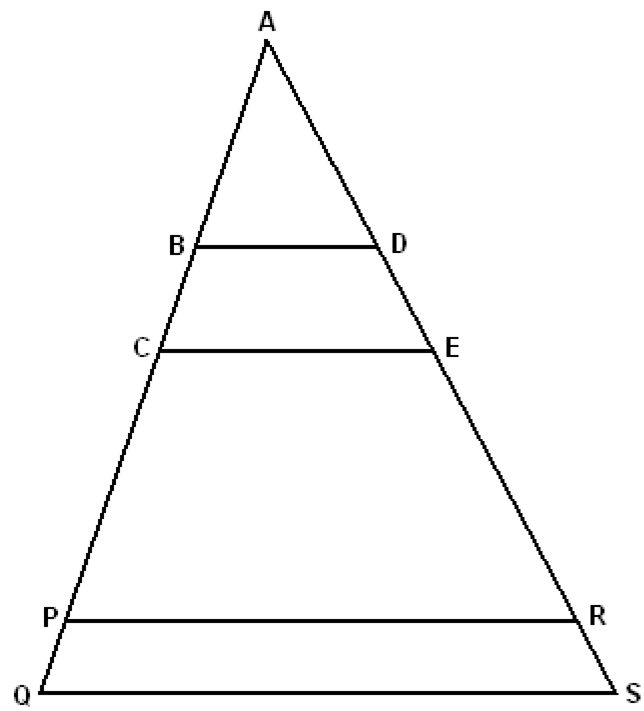
Two ratios of (positive real) numbers **a** / **b** and **c** / **d** are equal if and only if for each pair of positive integers m and n we have the following:

$$ma < nb \quad \text{implies} \quad mc < nd$$

$$ma > nb \quad \text{implies} \quad mc > nd$$

The derivation of this criterion is based upon a fundamentally important **rational density property** of the real numbers: *If we are given real numbers* **x** *and* **y** *such that* **x** < **y**, *then there is a rational number* **r** *such that* **x** < **r** < **y.** Further details about this implication are contained in the first supplement to this unit (see either history03a.pdf or histm003a.doc, which are different formats of the same document).

APPLICATION OF THE CONDITION OF EUDOXUS TO PROPORTIONALITY QUESTIONS. Suppose now that we have triangles **ABD** and **ACE** as in the figure below, where **BD** is parallel to **CE**; as in the figure we assume that the rays **[AB** and **[AC** are the same and likewise that the rays **[AD** and **[AE** are the same. Let **a** = |**AB**|, **b** = |**AC**|, **c** = |**AD**| and **d** = |**AE**|. We want to use the Condition of Eudoxus to conclude that **a** / **b** = **c** / **d.**



Suppose first that **m** and **n** are positive integers such that **ma < nb**. We want to show that **mc < nd**. We can find points **P** and **Q** on the ray **[AB = [AC** such that |**AP**| = **ma** and |**AQ**| = **nb**. Since **ma** < **nb**, it follows that **P** is between **A** and **Q**. One can then find unique parallel lines to **BD** and **CE** through **P** and **Q**. These lines will meet the line **AD** = **AE** in two points **R** and **S**. A proper formulation of concepts like betweenness, the two sides of a line, and so on will imply that **S** and **R** also lie on the ray **[AD = [AE** and that **R** is between **A** and **S**.

The proportionality results in the commensurable case now imply that

$$|\mathbf{AR}| \,/\, |\mathbf{AD}| \quad = \quad \mathbf{m} \quad = \quad |\mathbf{AP}| \,/\, |\mathbf{AB}| \qquad \text{and}$$

$$|\mathbf{AS}| \,/\, |\mathbf{AE}| \quad = \quad \mathbf{n} \quad = \quad |\mathbf{AQ}| \,/\, |\mathbf{AC}|.$$

Therefore |**AR**| = **mc** and |**AS**| = **nd** also hold. By observations in the previous paragraph we know that |**AR**| < |**AS**|, and thus we may use the preceding sentences to rewrite this as **mc** < **nd**. To summarize, we have now shown that **ma** < **nb** implies **mc** < **nd**.

If we have **ma < nb**, then we may proceed similarly. The argument is basically the same except that **Q** will be between **A** and **P**, and this will in turn imply that **S** is between **A** and **R**. Following the same line of reasoning in this case, one concludes that **ma** > **nb** implies **mc** > **nd**. Therefore we have established both parts of the Condition of Eudoxus, and consequently we have shown that **a/b = c/d**; by definition of the numbers **a, b, c, d** in this equation, the desired proportionality equation |**AB**| / |**AC**| = |**AD**| / |**AE**| is an immediate consequence.