3.D. The Platonic solids

The purpose of this addendum to the course notes is to provide more information about regular solid figures, which played an important role in Greek mathematics and philosophy.

We shall begin with comments on regular polygons in the plane. If we are given an arbitrary integer $\mathbf{n} \geq 3$ then a ***regular* n – *gon* P** is formed by starting with **n** vertex points $\mathbf{A_1, \ldots A_n}$ (it will be convenient to write $\mathbf{A_0}$ for $\mathbf{A_n}$ sometimes) such that

1. no three of the points are collinear
2. for each **k** between **1** and **n**, all vertices except $\mathbf{A_{k-1}}$ and $\mathbf{A_k}$ lie on the same side of the line $\mathbf{A_{k-1} A_k}$
3. for all **k** between **1** and **n**, the segments $\mathbf{[A_{k-1}A_k]}$ have the same length
4. for all **k** between **1** and **n – 1**, the angles $\angle \mathbf{A_{k-1}A_kA_{k+1}}$ have the same angular measurement as $\angle \mathbf{A_{n-1}A_nA_1}$

and taking **P** to be the union of the segments $\mathbf{[A_{k-1}A_k]}$. Given a positive number **s** it is always possible to construct a regular **n** – gon whose sides have length **s** (but not necessarily with straightedge and compass!), and it turns out that each vertex angle has a degree measurement given by

$$|\angle \mathbf{A_{n-1}A_nA_1}| \quad = \quad \mathbf{(n-2) \cdot 180^\circ / n}$$

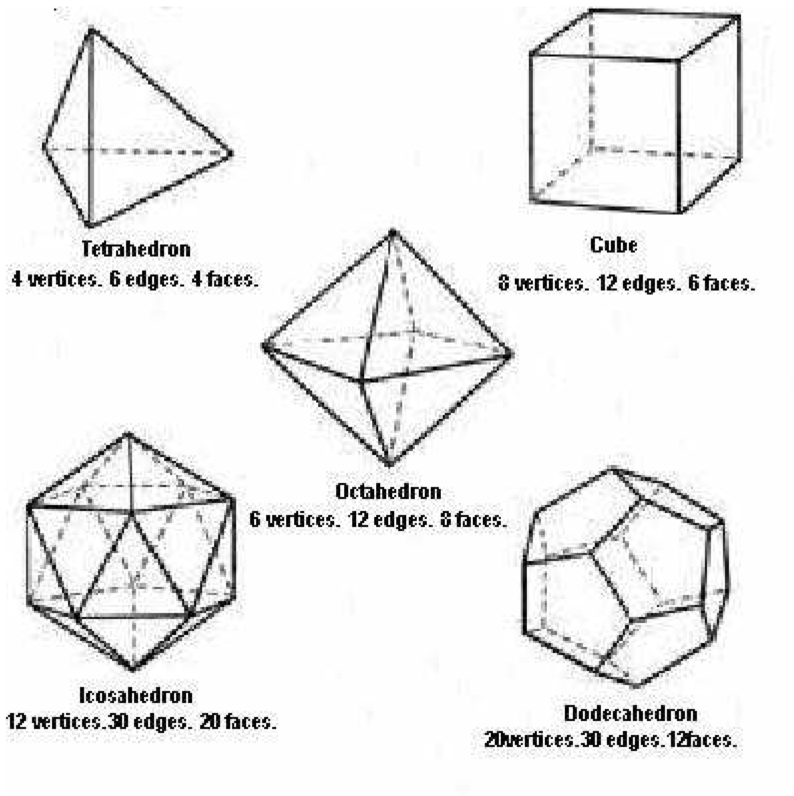
which specializes to 60° for equilateral triangles, 90° for squares, 108° for regular pentagons, 120° for regular hexagons, 145° for regular octagons, and so forth.

Given a regular polygon **P**, it is always possible to construct a **circumscribed circle,** which passes through every vertex **P** and also an **inscribed circle,** which is tangent to each segment $\mathbf{[A_{k-1}A_k]}$ of **P** at its midpoint. The perpendicular bisectors of the latter segments all meet at a single point which is the center of the two circles and also the center of mass for the region bounded by **P** (assuming uniform density).

Regular polygons beyond triangles and squares were known in prehistoric times, and in fact archaeologists have also discovered early examples of stones carved and marked to represent several (in fact, most) 3 – dimensional analogs of regular polygons. The two simplest examples are a cube and a triangular pyramid such that each face is an equilateral triangle. Each of these illustrates the fundamental properties that all regular solids should have.

1. Every 2 – dimensional face should be a regular **n** – gon for some fixed value of $\mathbf{n} \geq 3$.
2. Every 1-dimensional edge should lie on exactly two faces.
3. Every vertex should lie on **r** distinct faces for some fixed value of $\mathbf{r} \geq 3$.
4. No three vertices are collinear.
5. Given a face **F**, all vertices that are not on **F** lie on the same side of the plane containing **F**.

One noteworthy achievement of ancient Greek mathematics is the determination of all possible regular solid figures satisfying these conditions. In sharp contrast to the planar case, there are only finitely many possibilities that are illustrated below.



The Five Platonic Solids

(**Source:** http://www.goldenmeangauge.co.uk/platonic.htm)

Proving that there are only five regular solids requires a substantial amount of work and insight. Most proofs given today are based upon a formula due to Euler that was not mentioned in ancient Greek writings on regular solids: If the numbers of vertices, edges and faces for the solid are denoted by **v**, **e** and **f** respectively, then by Euler's formula we must have

$$\mathbf{v} + \mathbf{f} = \mathbf{e} + 2\,.$$

An analysis of this formula provides some insight into why there are only finitely many examples. The basic conditions on **n** and **r** imply that $2 \cdot \mathbf{e} = \mathbf{n} \cdot \mathbf{f} = \mathbf{r} \cdot \mathbf{v}$, and if one combines these with Euler's formula, the result is the following equation:

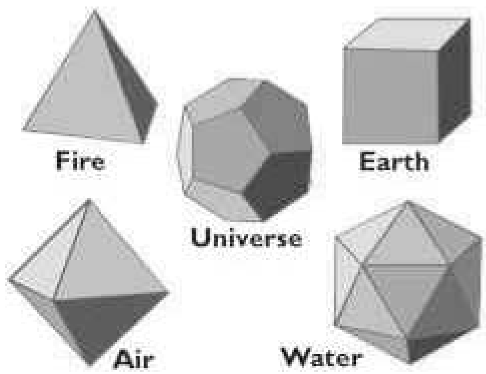
$$(1/\mathbf{n}) + (1/\mathbf{r}) - (1/\mathbf{e}) = \tfrac{1}{2}$$

There are not many values of **n** and **r** for which such an equation can hold, and in particular this can only happen if the sum of the reciprocals of **n** and **r** is strictly greater than ½ **.** The latter is true if and only if the pair **(n,r)** is given by one of (3,3), (3,4), (4,3), (3,5) or (5,3). These correspond to the regular tetradedron, the regular octahedron, the cube (regular hexahedron), the regular icosahedron and the regular dodecahedron; additional work is needed to verify the preceding assertion, but the use of Euler's formula is a fast and effective way of explaining informally why the number of possibilities is so limited.

As indicated in the second unit of the class notes, Plato theorized that these shapes represent the fundamental elements of nature. The following illustration from the online site

http://www.lsbu.ac.uk/water/platonic.html

summarizes his ideas in a simple graphic fashion.



Obviously one can pursue this much further, and there are many excellent online sites that describe further properties of the Platonic solids (*e.g.*, the existence and properties of inscribed and circumscribed spheres) and related objects studied by Archimedes and J. Kepler. Several also include further graphics, many of which are either interactive or animated. Here is a very incomplete list:

http://mathworld.wolfram.com/RegularPolyhedron.html

http://mathworld.wolfram.com/PlatonicSolid.html

http://www.math.utah.edu/~alfeld/math/polyhedra/polyhedra.html

http://www.mathacademy.com/pr/prime/articles/platsol/index.asp

http://www.enchantedlearning.com/math/geometry/solids/

http://www.dartmouth.edu/~matc/math5.geometry/unit6/unit6.html

http://www.mathsisfun.com/platonic_solids.html

http://www.mathpages.com/home/kmath096.htm

http://www.scienceu.com/geometry/facts/solids/

http://en.wikipedia.org/wiki/Platonic_solid

http://www.shef.ac.uk/nps/courses/groups/plato.html

http://math.youngzones.org/PlatonicSolids.html

http://astronomy.swin.edu.au/~pbourke/polyhedra/platonic/

http://whistleralley.com/polyhedra/platonic.htm

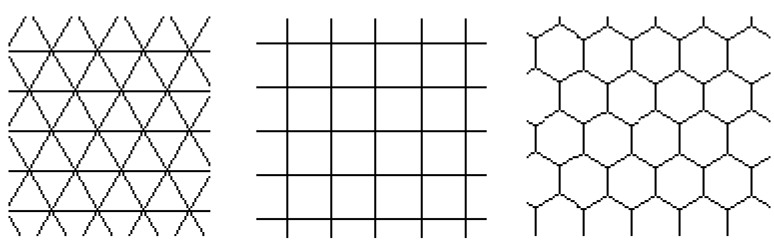
http://www.walter-fendt.de/m14e/platonsolids.htm

http://www.ul.ie/~cahird/polyhedronmode/photo.htm

Final remarks. We have noted that Book XIII of the Elements is devoted to the Platonic solids; the so-called Book XIV, which was most likely written by Hypsicles of Alexandria (190 – 120 B.C.E.), gives further results on this subject. For the sake of completeness we also note the existence of a so-called Book XV, which discusses further properties of regular solids and was written much later but is included along with Book XIV in many old versions of the *Elements*. In several respects it is inferior to Euclid's thirteen books and Hypsicles' contribution, and there is considerable disagreement about the authorship of this work and when it was first written. The commentary by Muhyi l'din al-Maghribi (1220 – 1283) is a particularly good reference for Book XV.

Regular tilings (tessellations) of the plane

The description of all regular solids is closely related to another basic geometrical question; namely, finding collections of regular polygons that cover the entire plane such that any two meet precisely on a common edge. If the regular polygons are squares then this corresponds to covering a flat surface by square tiles that do not overlap each other, and if the regular polygons are hexagons then this corresponds to the familiar honeycomb configuration of hexagons. A third example this type is the covering of a flat surface by tiles that are equilateral triangles. All of these are illustrated below.



Plane tilings or tessellations by triangles, squares and hexagons

Greek mathematicians (probably as early as the Pythagoreans) realized that the preceding examples were the only ones, and we shall prove this fact using ideas similar to those in the discussion of regular solids. Once again, two crucial numbers are the number **n** of sides for the regular polygons and the number **r** of polygons that contain a given vertex. In this case we do not have Euler's formula, but we have two other important conditions. One is the previously stated formula for the vertex angle measurements of a regular **n** – gon, and the other is the fact that the sums of all the angle measurements around a vertex must be 360 degrees. If we combine these we find that

$$\mathbf{360^{o} \quad = \quad r \cdot (n - 2) \cdot 180^{o} / n}$$

which is equivalent to

$$\mathbf{(1/n) \quad + \quad (1/r) \quad = \quad 1/2}$$

and the only solutions **(n,r)** for this equation are given by (3,6), (4,4) and (6,3). Thus the only possibilities involve equilateral triangles with 6 at each vertex, squares with 4 at each vertex and regular hexagons with 3 at each vertex.