4. Alexandrian mathematics after Euclid — I

(Burton, **4.1 – 4.5**)

During the 150 years after the death of Alexander the great, there were three mathematicians whose accomplishments in the subject were particularly outstanding. The first of these was Euclid, and in chronological order the others were Archimedes of Syracuse (287 – 212 B.C.E.) and Apollonius of Perga (262 – 190 B.C.E.). While Euclid is mainly known for his masterful exposition of earlier results by Greek and other mathematicians, both Archimedes and Apollonius are recognized for their deep and original work that went far beyond the findings of their predecessors in several respects. In particular, much of their research foreshadowed the development of analytic geometry and calculus more than 18 centuries later. Some of their major accomplishments will be summarized below.

Another extremely important contributor from this period was Eratosthenes of Cyrene (276 – 197 B.C.E.); there is an extensive summary of his work in Burton, and we shall add a few remarks and links to related web sites. The centuries after Alexander the Great were an extremely productive time for Greek mathematics, and many other talented individuals also made important contributions during this period. We shall mention a few of them at the end of this unit.

Archimedes

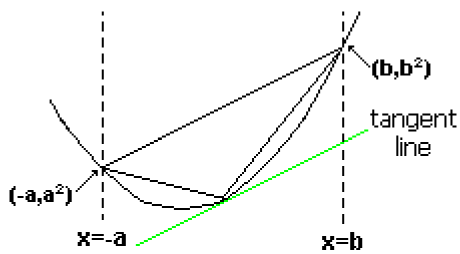
Archimedes is generally viewed as the most important contributor to mathematics during the classical Greek period (and even the period up to the Renaissance in Europe). This is due to the depth, insight, extent and originality of his work. It is only possible to mention a few of his contributions in a brief summary like these notes.

Although Archimedes and Euclid are two particularly outstanding figures in ancient Greek mathematics, the reasons are quite different. While Euclid is mainly known for presenting a large body of known mathematical results in a form that could be studied effectively, the writings of Archimedes are mainly devoted to new types of problems and new perspectives on old ones. His writings were addressed to individuals who had already mastered basic material, and as such they were less widely read or understood. This is probably one reason why Archimedes' work was preserved less systematically than Euclid's.

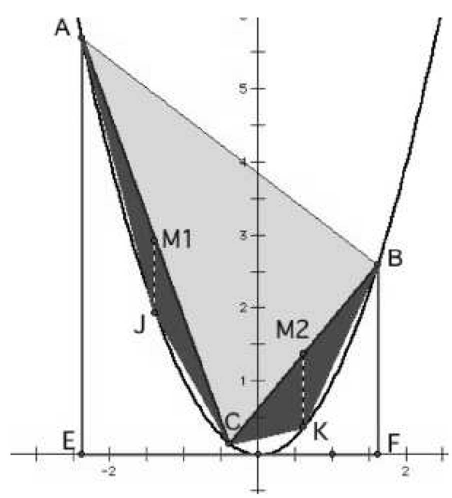
Here is a brief selective summary of some contributions to mathematics by Archimedes:

<u>Measurement of the the circle.</u> There is also a discussion of this in Burton from the bottom of page 191 to the end of the paragraph on page 195. The main contribution here is a proof that the area of a circle is $\boldsymbol{\pi}\, \boldsymbol{r}^2$, where $\boldsymbol{\pi}$ is the ratio of the circumference to the diameter (for whatever it might be worth, we should mention that the letter $\boldsymbol{\pi}$ was first used to denote this quantity in the early seventeenth century). Archimedes also estimates this number as lying between 3 + (10/71) and 3 + (1/7). This is done using the method of exhaustion developed by Eudoxus; similar but less logically sound ideas were previously advanced by the Sophist Antiphon (480 - 411 B.C.E.), who concluded that the actual values were reached for some figure with sides of some minimal infinitesimal length. The idea behind this is an example of the ***method of exhaustion*** first employed by Eudoxus; specifically, the region inside a circle is "exhausted" by taking an increasing sequence of inscribed regular polygons with 3×2^n sides for larger and larger values of ***n,*** and one examines the behavior of these measurements as ***n*** becomes increasingly large.

Quadrature of the parabola. The problem here is to find the area of a region bounded by a parabola and a segment joining two points on the parabola. This again uses the method of exhaustion and it is enlightening to compare his method with a more modern one that uses integral calculus at two key points. In particular, this provides some insight into just how close Archimedes came to discovering calculus 1900 years before Newton and Leibniz (and possibly what he missed). Archimedes proved that the area of the parabolic segment depicted below is 4/3 the area of the triangle inscribed in it according to the picture (specifically, the bottom vertex is situated so that its tangent line is parallel to the chord joining the other two points on the curve).



The first crucial step in his proof is an observation related to the picture below; namely, he shows that the combined areas of the smaller inscribed triangles **ACJ** and **BCK** are equal to one fourth the area of the larger triangle **ABC**. Thus the area of the inscribed polygon AJCKB is 5/4 the area of the area of the original triangle.



http://mtl.math.uiuc.edu/modules/module15/Unit%202/archim_ex.html

One can now apply the same argument to each of triangles **ACJ** and **BCK,** obtaining a third inscribed polygon whose area is

$$1 \;+\; (1/4) \;+\; (1/16)$$

times the area of the original triangle, and after doing this one can continue the process indefinitely. The site below has a nice animated picture of a few steps in the process:

http://www.ms.uky.edu/~carl/ma330/projects/parasegfin1.html

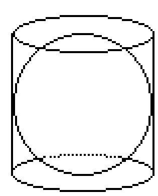
If we continue in this manner indefinitely we shall exhaust the entire region bounded by the parabola and the chord, and using infinite series we would conclude that the ration of the parabolic sector's area to that of the original triangle is equal to

$$\sum_{n=0}^{\infty} 4^{-n} = 1 + 4^{-1} + 4^{-2} + 4^{-3} + \cdots = \frac{4}{3}\,.$$

However, as noted in the discussion of Zeno's paradoxes, the Greek mathematicians did not work with infinite series, and therefore it was necessary for Archimedes to use another method (a double *reductio ad absurdum* proof) to verify that the ratio was indeed equal to this value.

In supplementary notes to this unit we shall indicate how one can retrieve Archimedes' result on the area of the parabolic sector using methods from integral calculus.

Sphere and cylinder. This is the work that Archimedes himself liked the best. Using a mixture of intuition based upon mechanical experiments, manipulating infinitesimals that he did not feel were logically adequate to use in formal proofs, and formal proofs themselves, Archimedes obtained the standard results we know today; for example, if one circumscribes a right circular cylinder about a sphere, the volume of the sphere is 2/3 the volume of the cylinder, and the surface of the sphere is 2/3 the total surface area of a circumscribed cylinder (including its bases). An illustration of this relationship became his epitaph.



Another important contribution of this work is a solution to the following problem: Given a number ***r*** between 0 and 1, determine where to cut a sphere by a plane so that the ratio of one part's volume to that of the entire sphere is equal to ***r.***

<u>The Method</u> **[*<u>of Mechanical Theorems</u>*]** ***.*** This manuscript, which was lost for 1500 years or more, unknown during the Middle Ages, and not discovered until 100 years ago, details how Archimedes obtained his results on areas and volumes by a mixture of mechanical experiments and logical deduction. He devised informal techniques using statics and infinitesimals to derive some conclusions that we would characterize today as integral calculus, but he then presented rigorous geometric proofs using the method of exhaustion for his results. This method of discovery is often described as a ***heuristic*** argument; for the sake of completeness we describe the latter more precisely:

> … something "providing aid in the direction of the solution of a problem but otherwise unjustified or incapable of justification." So heuristic arguments are used to show what we might later attempt to prove, or what we might expect to find in a computer run. They are, at best, educated guesses.

(**Source:** http://primes.utm.edu/glossary/page.php?sort=Heuristic)

Mathematicians and others have frequently used heuristic arguments to search for answers to questions before attempting to write down formal proofs, and it is difficult to imagine that this will ever change. In the words of Archimedes from the introduction to *The Method*, "It is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge."

The following link provides a good description of Archimedes' approach to measurement problems using infinitesimals:

http://en.wikipedia.org/wiki/How_Archimedes_used_infinitesimals

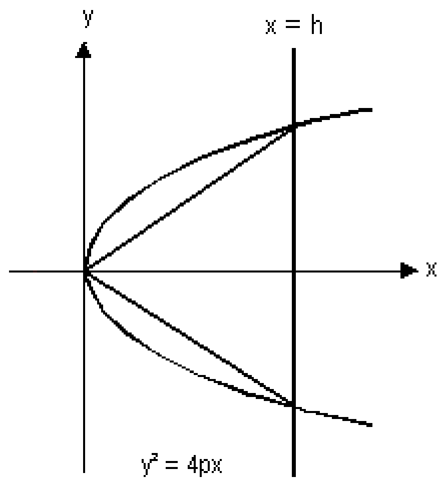
The importance of Archimedes' manuscript on *The Method* was immediately recognized upon its discovery. Unfortunately, for many years scholars had no access to the manuscript. In particular, it seems certain that modern technology would lead to the recovery of more information from the text than we currently have. Ownership of the manuscript changed hands in 1998, and the new owner has promised scholars access to this unique document, but apparently some issues still need to be resolved.

This and other works lead naturally to questions about how close Archimedes actually came to discovering integral calculus; however, the extent to which he actually discovered integral calculus nearly 2000 years before Newton and Leibniz is debatable.

Additional information on *The Method* appears on pages 198 – 199 of Burton.

Conoids and spheroids. This work calculates the area of the region bounded by an ellipse and volumes of some simple surfaces of revolution. In this book Archimedes calculates the areas and volumes of sections of cones, spheres and paraboloids. Here is one typical result: *Suppose that we are given a parabolic segment as in the picture below an inscribed isosceles triangle as illustrated. Let* **P** *and* **C** *be the respective solids of revolution formed by rotating the parabolic segment and triangle about the* **x**-*axis. Then the volume of* **C** *is two thirds the volume of* **P.**

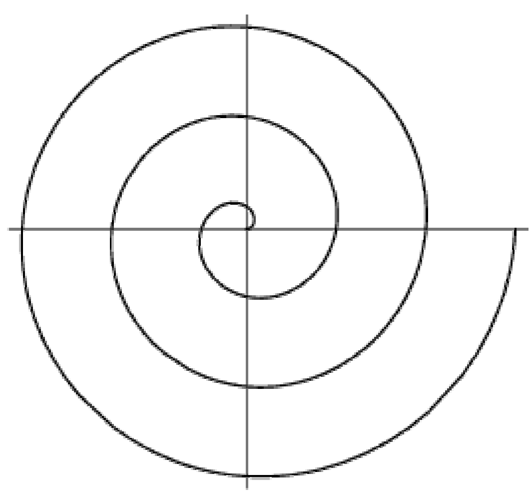
One can use calculus to derive such a formula fairly easily, particularly with a simplifying assumption such as p = h = 1 in the figure below.



Other results include volume formulas for hyperboloids of revolution and spheroids obtained by rotating an ellipse either about its major axis or about its minor axis.

<u>On spirals.</u> Since ancient times, many have viewed this as Archimedes' best and most remarkable work. He studies the properties of the curve given in polar coordinates by $r = \theta$ (the so-called ***Archimedean spiral***, also considered by Archimedes' close associate Conon of Samos, who lived from 280 B.C.E to 220 B.C.E.), and he proves a long list of facts about this curve. Many of these results are relatively straightforward once one has calculus at hand, and it is particularly striking to examine how Archimedes obtained all these results apparently without having calculus at his disposal. In particular, he gives results on tangent lines to the spiral as well as finding the areas of certain regions whose boundaries include pieces of the spiral.

Two important results in this work were the applications of the spiral to classical construction problems; namely, n – secting an angle for arbitrary $n > 2$ and squaring the circle. Additional information about Archimedes' results on the spiral appears on pages 196--197 of Burton and also in Exercises 10 – 12 on pages 200 – 201 of the latter.



The Sand Counter.
This was aimed at a more popular audience. The objective was to show how one could express and handle very large numbers like the number of grains of sand on a beach. In particular, this provided a method for describing numbers up to 10^{64}. Archimedes' approach to this problem anticipates the power notation that we use today for large numbers. A more detailed discussion of this work appears in Burton, beginning with the first new paragraph on page 195 and continuing through the first paragraph on the next page.

Equiponderance of planes. (also called ***On the Equilibrium of Planes***). This is more mechanics than mathematics, but it is relevant to both subjects because it develops the basic ideas involving the center of mass for a physical object. He was the first to identify the concept of center of gravity, and he found the centers of gravity of various geometric figures, assuming uniform density in their interiors, including triangles, paraboloids, and hemispheres. Using only the standard methods of Greek geometry, he also gave the equilibrium positions of floating sections of paraboloids as a function of their height [in his work ***On floating bodies***], a computation that is even a challenge for someone who has mastered first year calculus. This was probably an idealization of the shapes of ships' hulls. Some of his sections float with the base under water and the summit above water, which is reminiscent of the way icebergs float, although Archimedes probably wasn't thinking of this application.

The respect for Archimedes among ancient scholars is reflected in their use of the term *Archimedean Problem* to denote one that was exceptionally deep and difficult (cf. the term *Herculean task*) and *Archimedean proof* to denote an argument that was absolutely reliable and in the best possible form.

Biographical information about Archimedes

Some information about Archimedes' life is indisputable, but many aspects of the more colorful stories are questionable. He definitely had close ties to the rulers of his native city, Syracuse (Siracusa) in Sicily, and many things he did were for the benefit of the rulers of the city and the city itself. His resourcefulness and knowledge of mathematics and mechanics played an important role in the resistance that Syracuse mounted against Roman efforts at conquest, and it is universally accepted that he was killed when the Romans finally overran the city in 212 B.C.E., even though this was against the orders of the Roman general Marcellus who led the assault. On the other hand, ancient historians mention at least three possible ways in which the latter took place, and still other scenarios seem very plausible. The stories related to his best known scientific discovery – the Archimedean buoyancy principle for fluids – are also at least somewhat questionable, and in fact there are conflicting accounts for some details of these stories. Fortunately, the record of his scientific and engineering achievements is much more reliable than the personal anecdotes.