5. The late Greek period

(Burton, **5.1 – 5.4**)

During the period between 400 B.C.E. and 150 B.C.E., Greek mathematical knowledge had increased very substantially. Over the next few centuries, progress was more limited, and much of it involved mathematical topics like trigonometry that were needed in other subjects such as astronomy. However, before the end of the ancient Greek period during the sixth century A.D. there was some resurgence of activity that had important consequences for the future.

Diophantus of Alexandria

The known information about Diaphanous (conjecturally 200 – 284) is contained in a classic algebraic problem that is reproduced on page 207 of Burton. His most important work is contained in his *Arithmetica*, of which we now have the first six out of thirteen books; manuscripts claiming to be later books from this work have been discovered but their authenticity has not been established.

The *Arithmentica* of Diophantus differed greatly from earlier Greek writings in its treatment of purely algebraic problems in purely algebraic terms; as noted earlier, even the simplest algebraic equations had been analyzed in geometrical terms ever since the discovery of irrational numbers. Two aspects of *Arithmetica* are particularly noteworthy: One is his consideration of equations that have (usually infinitely) many solutions over the rational numbers or integers, and another is his introduction of special notation to manipulate mathematical relationships. Prior to this, mathematical writers stated and studied algebraic problems using ordinary language. In particular, an example of this sort from Egyptian mathematics is given on page 43 of Burton (see the discussion of Problem 24 from the Rhind Papyrus in the first paragraph of Section 2.3).

We shall describe Diophantus' notational innovations first. Mathematical historians generally classify algebraic notational systems into three types; namely ***rhetorical***, ***syncopated*** and ***symbolic***. The first of these corresponds to the original practice of stating things in terms of standard words and phrases, and the last corresponds to the use of letters and symbols that we use today. Syncopated notation is between these two, and although it does not use explicit symbols in the modern sense it adopts systematic abbreviations for basic concepts like unknowns and standard algebraic operations. This is the sort of notation that Diophantus used throughout his work. Several examples and more information appear on page 209 of Burton. Frequently mathematical histories characterize such notation as stenographic or shorthand, and either term is very descriptive.

Diophantus considers a fairly wide range of problems in his work, including some that have definite solutions and others that are *indeterminate*. Examples of the latter generally are systems of equations for which there are more unknowns than equations. In modern work we usually solve for some unknowns in terms of the others, but Diophantus was usually satisfied with finding just one solution of such equations.

However, he did insist that his solutions be positive rational numbers. His solutions and techniques are generally specialized and highly ingenious as opposed to systematic. One reason for this might be that despite his major notational innovations he still did not have the tools needed to formulate problems more generally. For example, his notation only allowed for one unknown; reducing an equation in several unknowns to a single unknown required clever insights and was done using words rather than his shorthand notation. He also lacked a symbol for a general number n.

Numerous examples are discussed in Section 5.2 of Burton and the accompanying exercises. The examples from the first two books include systems of first and second degree equations. Diophantus also seems to have been aware of some general number-theoretic patterns, although it is not clear whether he could prove them. Here are some examples:

1. A number of the form 4n + 2 or 4n + 3 cannot be written as a sum of two squares (or integers).
2. A number of the form 24n + 7 cannot be written as a sum of three squares.
3. **Every** positive integer can be written as a sum of at most four squares.

The first of these can be verified fairly directly, and the second is more challenging but can still be done in the same way, and it seems quite possible that Diophantus may have had proofs of these results. However, it is far less likely that he had a proof for the last statement. Fermat stated the result but could not prove it, and the first known proof was due to J. - L. Lagrange (1736 – 1813) in the late eighteenth century using results of Euler. Here are two online references:

http://planetmath.org/encyclopedia/LagrangesFourSquareTheorem.html

http://planetmath.org/encyclopedia/ProofOfLagrangesFourSquareTheorem.html

Even though modern mathematics usually has no problem in viewing irrational numbers as solutions to equations, there are both practical and theoretical situations in which one must or should have solutions of a more specialized type. For example, it is often necessary or useful to know whether a system of equations has a solution for which the values of some or all the unknown quantities are integers. When one uses terms like ***Diophantine equations*** or ***Diophantine problems*** today, it is generally understood that one is looking for solutions where the values of all the unknowns are integers.

Of course, specific Diophantine problems had been studied long before the work of Diophantus; for example, the study of integral solutions to the classical Pythagorean equation $x^2 + y^2 = z^2$ predated Greek mathematics by well over a thousand years. The so-called Cattle Problem attributed to Archimedes is discussed on pages 213 – 214 of Burton. It might be worthwhile to compare the description of the problem in Burton with the following translation of the Greek original:

http://www.mcs.drexel.edu/~crorres/Archimedes/Cattle/Statement.html

As noted in Burton, the solution of the cattle problem reduces to solving the Diophantine equation $x^2 - 4{,}729{,}494\, y^2 = 1$ where y is divisible by 9314. Not surprisingly, the integral solutions involve very large numbers, and a complete solutions was not obtained until the nineteen sixties with the help of computers; the solution obtained and confirmed at the time has over 200,000 digits.

Mathematicians from India and China were also interested in examples of Diophantine equations around the time of Diaphanous (say within two centuries or so of his work), and some aspects of their work are summarized on pages 214 – 219 of Burton.

The study of Diophantine equations continues to be a central topic in number theory. In general, it is difficult to determine whether a given Diophantine equation is solvable. For example, the Diophantine equation $x^2 - 94y^2 = 1$ is solvable, although the smallest solution is $x = 2{,}143{,}295$ and $y = 221{,}064$, but on the other hand the highly similar equation $x^2 - 94y^2 = -1$ has no solutions. Results from the middle of the twentieth century imply that there is no systematic way of deciding whether a given Diophantine equation is solvable. Here is an online reference:

http://www.ltn.lv/~podnieks/gt4.html

Diophantus refers to other writings of his that are now lost, and in particular he mentions the following result: Given any integers a and b then it is possible to find numbers c and d such that $a^3 - b^3 = c^3 + d^3$.

Pappus of Alexandria

Much of the late activity in Greek mathematics was devoted to summaries and commentaries on earlier work. This work is particularly important for mathematicians today because several of these commentaries survived to a great extent even though the original works are now lost. Pappus (290 – 350) was a particularly important contributor in this regard, for his writings indicate he had a very solid understanding of the earlier work and his own perspective on the earlier writings. His own results in geometry were the first major advances in centuries for that subject, and he is regarded as the last great mathmatician from the Hellenistic period.

Pappus' main (and best preserved) work was *The Collection* or *The Synagogue*, an extremely comprehensive treatise on geometry which included everything of interest to him. In several cases, he is our only source of knowledge about some mathematicians' work. Of the original eight books, only the first and part of the second are missing (and fortunately these are less crucial than the rest for modern scholarship). At many points in this work he added explanations, alternative approaches, and new results of his own. We have already mentioned his short and elegant proof of the Isosceles Triangle Theorem (see page 145 of Burton).

An extremely brief account of some results in Pappus' Synagogue appears on pages 221 – 222 of Burton, but a more extensive summary can be found at the following online site:

http://www.math.tamu.edu/%7Edallen/masters/Greek/pappus.pdf

Today Pappus may be best known for his results on the areas of surfaces of revolution and volumes of solids of revolution, which frequently appear in the text and exercises for standard calculus books. The result states that the area of a surface of revolution is the product of the length of the curve generating the surface times the distance traveled by the center of mass when rotated about the axis, and the volume of a solid of revolution is the product of the volume of the generating region times the distance traveled by its center of mass when rotated about the axis. This result was also published by P. Guldin (1577 – 1643) in the seventeenth century, and it is frequently known as the Pappus-Guldin Theorem. A discussion and partial derivation of the result appears in an addendum to these notes.

Some later commentators

This discussion will be limited to a few names. Theon of Alexandria (335 – 395) is particularly known for his edited version of Euclid's Elements that became standard. His daughter Hypatia (370 – 418) is the first prominent woman to appear in the history of mathematics, but all of her writings are lost. In many respects she marks the end of scholarly activity in Alexandria; at the end of the fourth century, Christianity became the state religion of the Roman Empire, and she was a victim of the resulting changes. Further discussion appears on pages 222 – 223 of Burton. We have already mentioned Proclus Diadochus as a valuable source about Greek mathematics during the period between Thales and Euclid. One further commentator who should be mentioned is Eutocius of Ascalon (480 – 540), whose recognition of Archimedes' work played an important role in preserving knowledge of the latter's contributions.

A combination of events is regarded as marking the end of ancient Greek mathematics; before these events, mathematical activities had slowed down or stopped altogether, and the main significance of the milestones is their irreversibility. We have already noted the adoption of Christianity as an official religion at the end of the fourth century, and this accelerated the decline substantially. The Platonic academy in Athens was finally closed during the early sixth century, and the Islamic conquests during the seventh century radically changed the basic culture in many places such as Egypt and Syria. Efforts to preserve the ancient Greek intellectual heritage continued throughout the existence of the Byzantine Empire until it ended in 1453, but this activity was extremely limited and entirely devoted to preservation rather than innovation. By the end of the Byzantine Empire intellectual activity in Western Europe had become revitalized and had begun to reacquaint itself with the work of ancient Greek mathematicians.

The Arab conquest of Alexandria

There is a widely circulated story about the destruction of the library in Alexandria after its conquest by the Caliph Omar (581–644; reigned 634 – 644) in 641, and it is summarized on page 223 of Burton. The discussion of this story in Burton is far more balanced than the comparable discussions in several other histories and mathematics texts, and in particular Burton mentions other periods during which the library appears to have been damaged seriously along with the deterioration of the library collection by the seventh century. More important, Burton's choice of words also suggests that details in the story may not be accurate.

In fact, there is only one source for this story, and it is a Christian writer who lived four centuries later. Other historical sources for the area during the seventh century say nothing about the matter, even in cases where one might expect to see some comments about such a major event and even if some parts of the story may be highly exaggerated (for example, fires from the burning books heating 4000 municipal baths for six months). The following links provide more detailed information:

http://www.bede.org.uk/library.htm

http://www.bede.org.uk/Library2.htm

http://www.ehistory.com/world/articles/ArticleView.cfm?AID=9

http://www.answers.com/topic/library-of-alexandria

http://www.newadvent.org/cathen/01303a.htm

A few comments seem necessary here. Although the historical records do not provide much evidence for the story about the library and no corroboration for the story that is frequently recounted, they cannot be used to disprove it either. Consequently, Burton's conjecture of some historical basis for the story may well be correct. There are many well documented instances of religious leaders ordering the destruction of valuable cultural artifacts; for example, one can point to the nearly complete destruction of Mayan writings by Bishop Diego de Landa in the sixteenth century and the more recent destruction of massive Buddhist sculptures in Afghanistan by the Taliban government in early 2001. Several issues deserve to be considered in connection with the story about the library in Alexandria.

1. As noted above, evidence in both directions is extremely limited and the central story is of questionable credibility.
2. For a century or more there had been little if any interest on anyone's part in whatever library contents may have remained after repeated destructions and prolonged disuse.
3. Islamic culture later played an extremely important role in the history of mathematics, and in fact many important mathematical writings of the Greeks are available to us today only because they were translated into Arabic.
4. At this point in history, the most unfortunate fact about the loss of ancient manuscripts is that they are no longer available to us.
5. Destructive or negligent actions by a wide range of individuals and cultures contributed significantly to this loss of ancient writings.
6. A great deal of ancient mathematical material survived in some form, and mathematics moved forward despite whatever might have happened.

The final point will lead directly to the next unit.