5.A. The Pappus Centroid Theorems

Although Pappus of Alexandria is known mainly for his very informed commentaries on the work of earlier Greek geometers, he also proved some original theorems. Here is an example:

Theorem. *Let A be a region in the upper half plane with boundary curve C, Let E be the solid of revolution formed by rotating A about the x-axis, and let F be the surface of revolution formed by rotating the boundary curve C about the same axis. Let a^* and c^* be the centers of mass for A and C respectively (assuming constant density). Then one has the following formulas:*

(1) *The volume of E is equal to the area of A multiplied by $2\pi\, a^*$.*

(2) *The surface area of F is equal to the length of C multiplied by $2\pi\, c^*$.*

From a physical viewpoint, both results are what one would expect, for they say that the mass of the solid/surface of revolution is the same as that of a ring which

(i) contains the centroid of the object,
(ii) has a center on the coordinate axis, and
(iii) has a constant density equal to the mass of the cross section A or C per unit length.

We shall derive special cases of Pappus' Theorem(s) using calculus. For the volume formula, we shall assume that A is bounded by the curves $x = f(y)$ on the right $x = g(y)$ on the left $y = p$ on the top, and $y = q \geq 0$ on the bottom. In general one can cut most figures up into pieces that each satisfy versions of this statement for suitable choices of boundary curves, but we shall not attempt to justify this here.

Proof of the volume formula. The Shell Method for computing volumes yields the formula

$$\mathrm{VOLUME}(E) \quad = \quad 2\pi \int_q^p \; y[f(y) - g(y)]\, dy$$

and one also has the basic forula

$$\mathrm{AREA}(A) \quad = \quad \int_q^p \; [f(y) - g(y)]\, dy \ .$$

On the other hand, the y-coordinate for the centroid of A is given by the formula

$$a^* \cdot \mathrm{AREA}(A) \quad = \quad \int_q^p \; y[f(y) - g(y)]\, dy$$

and if we substitute this into the first equation we obtain the conclusion of Pappus' Theorem for volumes.■

Proof of the area formula. In this case we think of the curve C as being given by $y = f(x) \geq 0$ for $p \leq x \leq q$. The standard formula for the area of the surface of revolution F is

$$\mathrm{AREA}(F) \quad = \quad 2\pi \int_q^p \; f(x)\sqrt{1 + f'(x)^2}\, dx$$

and the length of the curve is given by

$$\mathrm{LENGTH}(C) \quad = \quad \int_q^p \; \sqrt{1 + f'(x)^2}\, dx \ .$$

On the other hand, the y-coordinate for the centroid of C is given by the formula

$$c^* \cdot \mathrm{LENGTH}(C) \quad = \quad \int_q^p \; \sqrt{1 + f'(x)^2}\, dx$$

and if we substitute this into the first equation we obtain the conclusion of Pappus' Theorem for surface areas.■