6. Mathematics of Islamic and Hindu civilizations

(Burton, **5.5**, **6.1**)

As noted in the previous unit, Diophantine equations were also studied extensively in other ancient civilizations and particularly in India and China. In particular, for the following basic class of problems are given by a result known as the ***Chinese Remainder Theorem:***

Suppose that we are given two relatively prime positive integers **p** *and* **q***, and let* **n** *be a third positive integer. Suppose further that long division of* **n** *by* **p** *and* **q** *yields remainders of* **a** *and* **b** *respectively. What value(s) must* **n** *take?*

For example if **p** **=** 3 and **q** **=** 5, and the respective remainders are 2 and 3 respectively, then n must have the form 8 **+** 15**m** for some integer **m**.

Some of their results obtained by Chinese and Hindu mathematicians were similar to those of Diophantus, but others went much further. We shall pay particular attention to the contributions of Hindu mathematicians in these notes because their work had the greatest impact on mathematics as we know it today.

Mathematical activity in India

Indian mathematics has a long and interesting history. Although there may have been some mathematical interactions between Indian mathematics and Greek or Chinese mathematics, it is also clear that the Indian approach to the subject contained concepts and ideas that were not well developed by either of the other civilizations. In keeping with the focus of this course, we shall begin our discussion of mathematics around the time it developed ideas which ultimately had a major impact on modern mathematics.

Hindu or Indian mathematics is particularly known for developing the base ten numeration system we use today, with nine basic digits arranged in sequences and the roles of the digits determined by their placement. It is not entirely clear when this was developed, but the concept appears in writings of Aryabhatta the Elder (476 – 550), so we shall begin with his work.

The surviving mathematical work of Aryabhatta is contained in a manuscript called the *Aryabhattiya*, which is written entirely in verse and also covers other subjects besides mathematics. There is a passing reference describing a numbering system like the one we use today, and the mathematical portion of the work also contains results on integral solutions to Diophantine equations of the first and second degree. Trigonometry also played a significant role in Indian mathematics, and in fact modern mathematics follows the Indian approach – which is based upon the sine function – rather than the Greek approach, which was based upon the chord function **crd** discussed previously. The tables in the *Aryabhattiya* define trigonometric functions for angles with a basic increment of 3.75 degrees.

One of the most important figures in Indian mathematics was Brahmagupta (598 – 670), whose writings contain many important and far-reaching ideas. We shall list a few of them:

1. He explicitly recognized that Diophantine equations can have many solutions.
2. He used nine or ten symbols to write numbers.
3. He has no qualms about working with negative numbers and irrationals.
4. His work recognizes the concept of zero, although the first explicit use of a symbol for zero does not occur until late in the ninth century.
5. He devoted a great deal of effort to analyzing Diophantine equations like the so-called Pell equation $\mathbf{x^2 = 1 + a\,y^2}$. Further results on this equation due to Bhaskara (1114 – 1185) are mentioned later in this unit; Brahmagupta's main contribution was to give a method for constructing new solutions out of previously known ones.

Brahmagupta's writings also treat geometrical topics, but some of his conclusions are extremely inaccurate and far below the quality of his algebraic results. However, one particularly noteworthy geometric result due to him is an area formula for a quadrilateral that can be inscribed in a circle (see Exercise 6 on page 186 of Burton).

This is probably the most convenient time to say more about Bhaskara, who was also one of the most important figures in Indian mathematics. The concept of zero is far more explicit in his work, and he clearly understood that quadratic equations have two roots. His results on Pell's equation $\mathbf{px}^2 + 1 = \mathbf{y^2}$ include the following: When $\mathbf{p} = 61$ he found the solutions $\mathbf{x} = 226153980$, $\mathbf{y} = 1776319049$, and when $\mathbf{p} = 67$ he found the solutions $\mathbf{x} = 5967$, $\mathbf{y} = 48842$.

A great deal more could be said about the development of mathematics in India during the period considered above. In particular, Indian mathematicians were just as open to considering the concept of infinity as they were to working with zero. Furthermore, Indian mathematicians also obtained extensive results on infinite series; for example, Madhara (1340 – 1425) discovered the standard infinite series for **arctan x**, and subsequently others found an infinite series for $\pi/\mathbf{4}$ that converges much more rapidly than the standard series for **arctan** 1. However, since the most far-reaching consequences for modern mathematics were transmitted to the Western World indirectly through Arabic/Islamic civilization, we shall move on to the latter.

Arabic/Islamic mathematics

We have already mentioned that the term "Greek mathematics" refers to a fairly wide geographic area and contributions of mathematicians of many nationalities. The same can be said about the mathematics associated to Arabic and Islamic cultures over the period from about 800 to 1500, but the geographic area, the diversity of nationalities, and even the diversity of religions was even greater than in Greek mathematics. The geographic range included the entire Islamic world at the time, from Spain on the west to Uzbekistan on the east. In using phrases like Arab mathematics or Islamic mathematics it is important to remember the geographic, ethnic and religious diversity of those who worked within this framework.

Since the center of the Islamic world was, and still is, between Europe and the Indian subcontinent, it is not surprising that Arab/Islamic mathematics was heavily influenced by both Greek and Indian mathematics. For our purposes, two absolutely crucial legacies of Arab/Islamic mathematics are that it preserved a substantial amount of classical Greek mathematics that would otherwise have been lost or ignored, and it also passed along the important new insights that Indian mathematicians had discovered. However, Arab and Islamic mathematicians also made a number of important and highly original contributions, many of which were independently rediscovered by European mathematicians centuries later.

One of the best-known names in Arab mathematics was also one of the earliest: Abu Ja'far Muhammad ibn Musa **Al-Khwarizmi** (790 – 850). He is known for two major pieces of work. The first was an extremely influential account of the Hindu numeration system based upon an earlier Hindu work. Although the original Arabic versions of this work are lost, significantly altered Latin translations have survived, and typical Latin transliterations of his name as *Algoritmi* or *Algorism* have evolved into our modern word ***algorithm***. A second piece of his work has given us another basic word in mathematics. The word ***algebra*** comes directly from his book *Hisab* ***al-jabr*** *w'al-muqabala* (often translated with a phrase like "The Science of Restoration and Reduction"). An extended discussion of this work's contents appears on pages 227 – 232 of Burton, so we shall concentrate here on the nature of Al-Khwarizimi's contributions. There are significant differences of opinion about this. The problem solving methods in the work can be found in earlier writings of others, and the notation does not contain any significant advances. For example, everything is done using words, and there is no shorthand notation analogous to that of Diophantus. Irrational numbers are used freely as in Indian mathematics, but in contrast to the earlier work of Brahmagupta negative numbers are not considered. On the other hand, unlike the work of Diophantus, ***al-jabr*** takes a highly systematic approach to solving various sorts of equations (especially quadratic equations), with an emphasis on problem solving for its own sake rather than the theory of numbers. Even though there is no symbolism or shorthand along the lines of Diophantus, equations are frequently discussed using general terminology and words like *root* that have become standard mathematical vocabulary. Such discussions resemble modern verbal descriptions of problems, and from this perspective mathematical language appears to follow the pattern of most languages, with verbal formulations of concepts coming before an efficient symbolism is created for writing them down (another example mentioned earlier is the verbal discussion of zero in Brahmagupta's writings centuries before the first known symbolism for it). ***Al-jabr*** makes important progress towards removing extraneous geometrical ideas from solving equations (*e.g*, as found in Book II of Euclid's *Elements*), and as such plays an important role in separating these subjects from each other. However, the subjects are not completely uncoupled from each other, and in many instances Al-Khwarizimi uses geometrical ideas to prove that his algebraic solution techniques yield correct answers.

We have already mentioned Thabit ibn Qurra in our discussion of amicable pairs. His criterion for finding such pairs was completely original and has not been superseded by later results of others. He also translated many important Greek manuscripts into Arabic and made numerous other contributions to mathematics and other subjects; some further discussion of his mathematical work appears on page 233 of Burton.

Al-Khwarizimi's separation of algebra from geometry was completed by Abu Bekr ibn Muhammad ibn al-Husayn **Al-Karaji** (953 – 1029), who also took important steps in defining nonzero integral powers (including negative ones!) algebraically and came very close to discovering the law of exponents $\mathbf{x^n\, x^m = x^{m+n}}$ where **m** and **n** are integers; the only missing element was that he did not set $\mathbf{x^0}$ equal to 1. Al-Karaji also used a partially developed form of mathematical induction in some of his proofs, and in particular in an argument essentially showing that

$$1^3 + \ldots + \mathbf{n}^3 \quad = \quad (1 + \ldots + \mathbf{n})^2.$$

Many Arabic/Islamic mathematicians worked extensively on computing values of trigonometric functions because of their importance in astronomy. We shall only mention one name specifically here: Mohammad **Abu'l-Wafa** Al-Buzjani (940 – 998). Among other things, he worked extensively with all six of the basic trigonometric functions in use to day and invented the secant and cosecant functions. He also devised new methods for computing sines of angles, and compiled tables of values with incremental intervals of 0.25 degrees; in modern decimal notation his results were accurate to 8 decimal places. By contrast, the tables of Claudius Ptolemy were accurate to three decimal places in modern notation (at the time values were expressed in Babylonian sexagesimal form, a practice which continued for a few centuries longer). Abu'l-Wafa is also known for his writings on geometry, which discuss at length the repeating, abstract geometric patterns that play an important role in Islamic art and architecture. The following online reference summarizes these geometrical writings:

http://www.mi.sanu.ac.yu/vismath/sarhangi/

Normally we associate the name Omar Khayyam (1048-1122) with the poetic work *Rubaiyat*, and its nineteenth century English translation by E. Fitzgerald, but Khayyam also made a few significant contributions to mathematics. We shall only mention one of them here. Recall that the Greek mathematician Manaechmus solved the problem of duplicating a cube by using two intersecting parabolas to construct a segment whose length was **cbrt**(2). Khayyam gave methods for geometrically finding roots of more general cubic equations by constructing various conic sections and taking their intersection points. For example, in one type of problem the root is given by intersecting a circle and a parabola. This case is discussed in detail on pages 234 – 235 of Burton, and two others are presented in Exercise 7 on page 243 of that text.