7. Mathematical revival in Western Europe

(Burton, **6.2 – 6.4**, **7.1**)

Although mathematical studies and discoveries during the Dark Ages in Europe were extremely limited, there were contributors to the subject during the period from the Latin commentator Boëthius (475 – 524) to the end of the 12^{th} century. Several names are mentioned in Sections 5.4 and 6.1 of Burton, and the latter's exercises also mention Alcuin of York (735 – 804) and Gerbert d'Aurillac (940 – 1003), who became Pope Sylvester II (997 – 1003),.

During the second half of the 11^{th} century some important political developments helped raise European's consciousness of ancient Greek mathematical work and the more recent advances by Hindu and Islamic mathematicians. Many of these involved Christian conquests of territory that had been in Moslem hands for long periods of time. Specific examples of particular importance for mathematics were the Norman reconquest of Sicily, the Spanish ***reconquista*** during which extensive and important territories in the Iberian Peninsula changed hands, and the start of the Crusades. From a mathematical perspective, one important consequence was dramatically increased access to the work of Islamic mathematicians and their translations of ancient Greek manuscripts. Efforts to translate these manuscripts into Latin continued throughout the 12^{th} century; the quality of these translations was uneven for several reasons (for example, in many the Arabic manuscripts were themselves translations from Greek), but this was an important step to promoting mathematical activity in Europe. A more detailed account of this so-called *Century of Translation* appears on pages 253 – 257 of Burton.

Fibonacci

Leonardo of Pisa, more frequently known as Fibonacci (1170 – 1250) symbolizes the revival of mathematical activity in Europe during the late Middle Ages, and his book *Liber abaci* (Book of Counting – literally, the abacus), which appeared in 1202, is the first major work aimed specifically at a European audience that recounts some important ideas from Hindu and Islamic mathematics and integrates this work with that of earlier contributions from Greek mathematics. The work is not merely a routine compilation of material from other sources, but rather it represents an independent and broadly based point of view.

Despite the impact of *Liber abaci* during the late Middle Ages, the first printed version did not appear until 1857, nearly 650 years after it was first written; the first English translation appeared in 2003.

Comments on the contents of Fibonacci's writings

Certainly the most far reaching aspect of *Liber Abaci* is its presentation of the Hindu-Arabic number system and the large amount of evidence it produces to demonstrate the superiority of the Hindu-Arabic notation and Hindu methods of computation. However, there are several other noteworthy features. Some ideas in the book were very advanced for that time, but many aspects of the notation are clumsy by modern standards.

As Burton notes, it is somewhat ironic that today Fibonacci is best known for one problem from his book that was named after him in 1877 by E. Lucas (1842 – 1891); indeed this sequence appears in writings of the Indian mathematicians Hemachandra (1089 – 1173) and Gospala around 1135, and it also appears in seventh century Indian writings.

$$F_n = \frac{(1+\sqrt{5}\,)^n - (1-\sqrt{5}\,)^n}{2^n\sqrt{5}}.$$

In *Liber quadratorum* Fibonacci investigates several ***number-theoretic questions involving perfect squares***. Much of this is described on pages 260 – 261 of Burton. As noted there, one basic topic of interest in this book is the following:

Given an arithmetic progression of integers **a + n d** (*where* **a** *and* **d** *are fixed integers and* **n** *is the variable*), *can one find three or more successive values that are perfect squares?*

Burton gives one example of such a triple – namely, 1, 25 and 49 – for which the difference **d** is equal to 24, and some of the exercises discuss additional aspects of this question. Remarks in Burton suggest that other triples of this type exist, but nothing further is stated, so we shall fill in some details here. In fact, there are 67 sequences of three elements whose constant difference was 10,000 or less. The next few are given as follows:

$\{2^2, 10^2, 14^2\}$, constant difference = 96
$\{7^2, 13^2, 17^2\}$, constant difference = 120
$\{3^2, 15^2, 21^2\}$, constant difference = 216
$\{7^2, 17^2, 23^2\}$, constant difference = 240

Fibonacci actually studied this question of squares in an arithmetic progression extensively, and in particular he completely characterized the common differences **d** that can arise from consecutive triples of perfect squares. He called these differences ***congruous numbers***, and they are defined in the second part of Exercise 5 on page 265 of Burton. In this terminology, the objective of Exercise 5(b) is to show that every congruous number is divisible by 24. We shall not give a proof of Fibonacci's result relating congruous numbers to consecutive squares in arithmetic progressions, but here is a proof that the common difference d is always divisible by 24; it is taken from the following online site:

http://nrich.maths.org/askedNRICH/edited/3412.html

Suppose $\mathbf{a}^2$, $\mathbf{b}^2$ and $\mathbf{c}^2$ are in an arithmetic sequence whose constant difference isn't a multiple of 8. Without loss of generality we might as well assume that **G.C.D. (a,b) = G.C.D. (b,c) = 1.** Note that **a** and **c** cannot both be even since then $\mathbf{b}^2 = (\mathbf{a}^2 + \mathbf{c}^2) / 2$ is even so **a** and **b** would share a factor of 2. This means that at most one of **a, b ,c** is even.

Now if **x** is an odd integer then it is easily checked that $\mathbf{x}^2 = 1 \pmod 8$ [by one of the exercises, if **x** is odd we know that **x** leaves a remainder of 1 when divided by 8] and if **x** is even then either $\mathbf{x}^2 = 0 \pmod 8$ or $\mathbf{x}^2 = 4 \pmod 8$. [The notation means that both sides of the equation have the same remainder when divided by 8]. Therefore working mod 8 the triple $(\mathbf{a}^2, \mathbf{b}^2, \mathbf{c}^2)$ must be one of the following:

(0,1,1)
(4,1,1)
(1,0,1)
(1,4,1)
(1,1,0)
(1,1,4)

It is clear that none of these are in arithmetic sequence, so this is a contradiction establishing that the common difference must be a multiple of 8.

Now if the arithmetic progression has common difference $\mathbf{d} = 1 \pmod 3$ then $\mathbf{a}^2, \mathbf{b}^2, \mathbf{c}^2$ must each have distinct [remainders or] residues (mod 3); in particular one of them must be equal to $-1 \pmod 3$ which is impossible. Likewise if the common difference $\mathbf{d} = -1$ (mod 3) then $\mathbf{a}^2, \mathbf{b}^2, \mathbf{c}^2$ have distinct residues (mod 3) which leads to the same contradiction. So we must have $\mathbf{d} = 0$ (mod 3). Therefore the common difference is a multiple of 8 and 3, and hence it must be a multiple of 24.

One can also ask whether there are even longer sequences of squares in arithmetic progressions, and a result of P. de Fermat (1601 – 1665) and L. Euler states that no such sequences exist. Here is a simple consequence of the nonexistence result which is taken from the following online source:

http://www.mathpages.com/home/kmath044.htm

H. Jurjus has asked whether there are rational numbers **p**, **q** such that $(\mathbf{p}^2, \mathbf{q}^2)$ is a point on the hyperbola given by

$$(2 - x)(2 - y) = 1$$

with $(\mathbf{p}^2, \mathbf{q}^2)$ not equal to (1,1). The answer is no. To see why, suppose we have rational numbers $\mathbf{p} = \mathbf{a} / \mathbf{b}$ and $\mathbf{q} = \mathbf{c} / \mathbf{d}$ (both fractions reduced to least terms). Then if $(\mathbf{p}^2, \mathbf{q}^2)$ is on the hyperbola we have

$$(2\mathbf{b}^2 - \mathbf{a}^2)(2\mathbf{d}^2 - \mathbf{c}^2) = \mathbf{b}^2 \mathbf{d}^2$$

Since our fractions are reduced to least terms, it follows that $\mathbf{b}^2$ is coprime to $2\mathbf{b}^2 - \mathbf{a}^2$ [no common divisors except 1] and $\mathbf{c}^2$ is coprime to $2\mathbf{d}^2 - \mathbf{c}^2$ so that

$$\mathbf{b}^2 = 2\mathbf{d}^2 - \mathbf{c}^2 \quad \text{and} \quad \mathbf{d}^2 = 2\mathbf{b}^2 - \mathbf{a}^2.$$

Rearranging terms we see that

$$\mathbf{b}^2 - \mathbf{d}^2 = \mathbf{d}^2 - \mathbf{c}^2 \quad \text{and} \quad \mathbf{d}^2 - \mathbf{b}^2 = \mathbf{b}^2 - \mathbf{a}^2 .$$

Together these equations imply that $\mathbf{a}^2, \mathbf{b}^2, \mathbf{d}^2$ and $\mathbf{c}^2$ are in arithmetic progression, which is impossible.

Another noteworthy achievement of Fibonacci was his ***solution of the cubic equation***

$$\mathbf{x}^3 + 2\mathbf{x}^2 + 10\mathbf{x} = 20$$

which was reportedly given to him as a challenge. His numerical approximation to the root is described on page 262 of Burton; aside from the accuracy of the result, it is worth noting how, despite his writings on the Hindu-Arabic numeration system, he (and others, including Arabic mathematicians) still wrote fractional values in the Babylonian sexagesimal notation). In his analysis of this equation he also made an important observation which foreshadowed the nineteenth century results on the impossibility of trisecting angles and duplicating cubes by means of straightedge and compass. It appears that the original version of the problem was to find a roof of the given cubic equation by means of classical Greek straightedge and compass methods. Fibonacci proved that the root could not be obtained by such methods.

Rather than attempt to give Fibonacci's proof, we shall analyze the equation from the same viewpoint we employed to study the construction problems. As in those cases, the proof that a root cannot be found using straightedge and compass depends upon showing that the polynomial $\mathbf{x}^3 + 2\mathbf{x}^2 + 10\mathbf{x} - 20$ cannot be factored into a product of two rational polynomials of lower degree, or equivalently the results of Gauss, it does not have an integral factorization of this sort. If it had such a factorization then it would have a linear factor and hence an integral root. Furthermore, if it had an integral root then this root would have to divide 20 and thus would have to be one of the following:

$$\pm 1, \quad \pm 2, \quad \pm 4, \quad \pm 5, \quad \pm 10, \quad \pm 20$$

One is then left to check that none of these twelve integers is a root of the given polynomial. Direct substitution is perhaps the most immediate way of attacking this problem, but one can also dispose of many possibilities simultaneously by noting that the polynomial under consideration is positive for all integers $x > 1$ and it is negative for all integers $x < 0$.

Fibonacci's writings on Pythagorean triples are discussed in Section 6.4 of Burton (pages 273 – 277).

Jordanus Nemorarius

One noteworthy contemporary of Fibonacci was Jordanus Nemorarius (1225 – 1260), who made contributions in several areas. He successfully analyzed mechanical problems about inclined planes that Archimedes had not been able to solve, and he was the first western mathematician to use letters consistently as symbols for unknown quantities. However, aside from this innovation his mathematical terminology was rhetorical (see pages 263 – 264 of Burton).

One result of Jordanus proved a basic relationship between perfect and nonperfect numbers. Let us define a positive number to be abundant if it is less than the sum of its proper divisors and deficient if it is greater less than the sum of its proper divisors. The result of Jordanus states that a nontrivial multiple of a perfect number is abundant and a nontrivial divisor of a perfect number is deficient. In particular, this implies that every nontrivial multiple of 6, 28, … is abundant.

Nicole Oresme

On page 264 of Burton there is brief reference to Nicole Oresme (1323 – 1382; there is a misprint in Burton). Oresme made several highly original contributions to mathematics, many of which were centuries ahead of their time and some of which had a more immediate impact.

Fractional exponents. Oresme proposed a mathematically sound way of defining positive fractional powers of a number and even raised the possibility of irrational powers like **sqrt**(2).

Graphical representation of functions. The book, *Tractatus de figuratione potentiarum et mensurarum* (Latitude of Forms), written by Oresme or one of his students, popularized the idea of representing variable quantities graphically; we have already noted that the methods of Apollonius had anticipated the development of coordinate geometry much earlier, but the idea of representing variables was presented very clearly in Oresme's work, and its influence can be measured by the numerous editions of his work that were published well into the sixteenth century. His suggestion that physically measurable quantities are continuous has been implicitly assumed in many applications of mathematics to science and engineering for centuries. Oresme also speculated about graphical representations of quantities dependent on two variables by surfaces in three dimensions and possibly about analogs in even higher dimensions, but the mathematical notation at the time was inadequate.

Infinite series. During the fourteenth century western mathematicians began to cast aside the Greek reluctance to consider infinite processes, and in particular various infinite series were studied. It should be noted that Indian and Chinese mathematicians had studied such series much earlier, and some particularly noteworthy results of theirs from earlier times through the fifteenth century were rediscovered after the relevance of calculus to infinite series became apparent in the early eighteenth century. In particular, the Indian mathematician Madhara (1340 – 1425) discovered the familiar infinite series for the inverse tangent function and the specialization to an infinite series for $\pi/4$, and in the next century scholars continuing his work discovered a series that converges far more rapidly:

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \ldots$$

Since geometric series are probably the simplest and most basic examples, it is not surprising that medieval mathematicians were able to derive the standard formulas for such series without much trouble, and in fact they looked at numerous other problems.

Oresme used his graphical approach to provide an elegant proof for the following infinite series formula due to Robert Suiseth (or Swineshead) (ca. 1340-1360), better known as Calculator:

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots = 2.$$

Using results on rearrangements of series one can derive this result using modern methods by looking at the following tableau:

$$\begin{aligned}
&\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots \\
&\qquad = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\
&\qquad\qquad + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\
&\qquad\qquad\qquad + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\
&\qquad\qquad\qquad + \cdots \\
&\qquad = 1 + \frac{1}{2}(1) + \frac{1}{4}(1) + \cdots \\
&\qquad = 2
\end{aligned}$$

Note that in ordinary addition of finite sums, the answer does not depend upon the order or grouping of summation and that the regrouping suggested by the preceding tableau involves an infinite rearrangement and regrouping. For sums of positive quantities it is possible to justify such rearrangements, but if one is working with sums that have both positive and negative terms, then serious problems can arise. In particular, if we evaluate the infinite series for the inverse tangent of $\boldsymbol{x}$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

at $\boldsymbol{x} = 1$ we know that the answer is $\pi/4$, but we can rearrange the terms in this series to realize any real number as the sum. A reference for this fact is W. Rudin, ***Principles of Mathematical Analysis*** (3rd Ed.), pages 75 – 77.

Finally, Oresme appears to be the first person in the history of mathematics to discover that the harmonic series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots+\frac{1}{n}+\cdots$$

diverges. His proof is the same one that is often seen today in textbooks: The sum of the first term by itself is ½, the sum of the next two terms is also ½, the sum of the next four terms is again ½, and so on, so that if one adds together sufficiently many terms from this series the sum will exceed any chosen positive real number.

Addendum A. *The closed formula for Fibonacci numbers*

We shall give a derivation of the closed formula for $\mathbf{F_n}$ here. This formula is known as Binet's formula because it was derived and published by J. Binet (1786 – 1856) in 1843. However, the result had been known to several well known mathematicians including L. Euler (1707 – 1783), Daniel Bernoulli (1700 – 1782) and A. De Moivre (1667 – 1754) more than a century earlier.

We begin with general observations. Suppose we are given a sequence defined recursively by a formula

$$\mathbf{x_n} = \boldsymbol{b}\,\mathbf{x_{n-1}} + \boldsymbol{c}\,\mathbf{x_{n-2}}$$

where **b** and **c** are real numbers. The first point is that such a sequence is uniquely determined by $\mathbf{x_0}$ and $\mathbf{x_1}$. Suppose that $\mathbf{x_n}$ and $\mathbf{y_n}$ are two such sequences with values of **P** and **Q** when **n** = 0 or 1. Then $\mathbf{x_n} = \mathbf{y_n}$ when **n** = 0 or 1; assume that the values of the sequence are equal for all $\mathbf{n} < \mathbf{k}$, where $\mathbf{k} > 1$. Then we have

$$\mathbf{x_k} = \boldsymbol{b}\,\mathbf{x_{k-1}} + \boldsymbol{c}\,\mathbf{x_{k-2}} = \boldsymbol{b}\,\mathbf{y_{k-1}} + \boldsymbol{c}\,\mathbf{y_{k-2}} = \mathbf{y_k}$$

and therefore the two sequences are equal by mathematical induction.

In favorable cases one can write down the sequence $\mathbf{x_n}$ in a simple and explicit form. Here is the key step.

PROPOSITION. *Suppose that* **r** *and* **s** *are distinct roots of the* ***auxiliary polynomial***

$$\mathbf{t^2 - b\,t - c}\,.$$

Then for every pair of constants ***u, v*** *the sequence*

$$\boldsymbol{u}\,\mathbf{r^n} + \boldsymbol{v}\,\mathbf{s^n}$$

solves the equation $\mathbf{x_n} = \boldsymbol{b}\,\mathbf{x_{n-1}} + \boldsymbol{c}\,\mathbf{x_{n-2}}$.

Derivation. Let $\mathbf{y_n} = \boldsymbol{u}\,\mathbf{r^n} + \boldsymbol{v}\,\mathbf{s^n}$; we need to show that

$$\mathbf{y_n} - \boldsymbol{b}\,\mathbf{y_{n-1}} + \boldsymbol{c}\,\mathbf{y_{n-2}} = 0$$

for $\mathbf{n} > 1$. Let's expand the left hand side and see what we get.

$$(ur^n + vs^n) - b(ur^{n-1} + vs^{n-1}) - c(ur^{n-2} + vs^{n-2}) =$$

$$u(r^n - br^{n-1} + cr^{n-2}) + v(s^n - bs^{n-1} + cs^{n-2}) =$$

$$ur^{n-2}(r^2 - br + c) + vs^{n-2}(s^2 - bs + c) =$$

$$ur^{n-2}\cdot 0 + vs^{n-2}\cdot 0 = 0.$$

Therefore $ur^n + vs^n$ solves the original equation.

One can take this further to find the unique solutions satisfying $x_0 = P$ and $x_1 = Q$ by solving the equations $u + v = P$ and $ur + vs = Q$ for u and v. One can always find a unique solution because **r** and **s** are distinct.

We now specialize all this to the Fibonacci equation

$$F_n = F_{n-1} + F_{n-2}$$

whose auxiliary polynomial is equal to

$$x^2 = x + 1.$$

Equivalently, one can write this polynomial in the form

$$x^2 - x - 1 = 0$$

and since the roots of this equation are

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2}$$

it follow that the closed formula for the Fibonacci sequence must be of the form

$$f_n = u\phi^n + v\psi^n$$

for some constants u and v. If we use the conditions $F_0 = 0$ and $F_1 = 1$ we see that

$$0 = u\phi^0 + v\psi^0, \qquad 1 = u\phi^1 + v\psi^1$$

where the first equation simplifies to $u = -v$; substituting this into the second one yields

$$1 = u\left(\frac{1+\sqrt{5}}{2}\right) - u\left(\frac{1-\sqrt{5}}{2}\right) = u\left(\frac{2\sqrt{5}}{2}\right) = u\sqrt{5}.$$

Therefore

$$u = \frac{1}{\sqrt{5}}, \qquad v = \frac{-1}{\sqrt{5}}$$

and accordingly we have

$$f_n = \frac{\phi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$