8. The sixteenth century: Cubic and quartic formulas

9. Mathematics and the Renaissance

(Burton, **7.1 – 7.4, 8.1**)

The transition from period began with the breakthroughs of Indian mathematicians and continued with the work of Arabic/Islamic mathematics. Both of the latter remained productive through much of the second half of this transitional period, which roughly covers the time from 1200 to 1600. In the preceding unit we discussed the beginning of the second half of the transitional period, during which there was a revival of activity in Christian Western Europe. We are combining the two units listed above because the events described in Unit 8 took place towards the end of the period covered in Unit 9.

General remarks about late medieval and Renaissance mathematics

Many histories of mathematics view the time after Fibonacci until the early sixteenth century as a period of decline and inactivity. While the work during that period did not contain any advances at the level of Fibonacci's work, there were some important developments involving mathematics that took place during that time. Many can be placed into the following two categories:

1. Improvements in mathematical notation
2. Stronger ties to science, the arts and commerce

The Italian teachers

As Italian merchants developed more extensive and complex trading relationships with other nations after the Crusades, their needs to understand and work with mathematics increased significantly. This led to the emergence of a new class of mathematicians, who wrote texts from which they taught the necessary material. Large numbers of such texts have been preserved. These Italian mathematicians of the 14th century were instrumental in teaching merchants the ``new'' Hindu-Arabic decimal place-value system and the algorithms for using it. There was formidable resistance to the new numbering system and computational techniques, both in Italy and the rest of Europe, but eventually the new methods, which were of course more efficient and convenient to use, became the accepted standard, first in Italy and later throughout the rest of Europe.

The Italians were thoroughly familiar with Arabic/Islamic mathematics and its emphasis on algebraic methods. Although their teaching focused on practical business problems they also studied various recreational problems, including examples in geometry, elementary number theory, the calendar, and astrology. In connection with their instructional and recreational mathematics, they extended the Islamic methods by introducing *abbreviations and symbolisms*, developing new methods for dealing with complex algebraic problems and allowing the use of symbols for unknowns. Thus, unlike Islamic algebra, which was entirely rhetorical, the algebra of the Italians generally used syncopated notation to varying degrees. Another innovation was the extension of the Arabic/Islamic techniques for solving quadratic equations to higher degree polynomials. For example, near the middle of the 14th century Maestro Dardi of Pisa extended al-Khwarizimi's standard list of 6 types of quadratic equations to a list of 198 equations of degree less than or equal to 4, and he gave a method for solving one type of cubic equation. .

Here is an example of one problem from this era, which was formulated by Antonio de Mazzinhi (1353 – 1383): *Find two numbers such that multiplying one by the other yields* 8 *and the sum of their squares is* 27*.*

The solution begins by supposing that the first number is one number minus the root of some other number, while the second number equals one number minus the root of some other number. The problem leads to the equations

$$\begin{aligned}(x-\sqrt{y})(x+\sqrt{y}) &= 8\\ (x-\sqrt{y})^2+(x+\sqrt{y})^2 &= 27\end{aligned}$$

for which the solution is given by

$$x=\frac{\sqrt{43}}{2},\quad y=\frac{11}{4}\ .$$

.

The mathematical theory of perspective drawing

As the cultural and commercial center of the late Middle Ages and early Renaissance, Italy was the source of many important new developments during that period. An important change took place in painting around the year 1300. Prior to that time the central objects of paintings were generally flat and more symbolic than real in appearance; emphasis was on depicting religious or spiritual truths rather than the real, physical world. As society in Italy became more sophisticated, there was an increased interest in using art to depict a wider range of themes and to so in a manner that more accurately captured the image that the human eye actually sees.

The earlier concepts of art are clearly represented in a segment from the famous Bayeux Tapestry which is a graphic account of the Norman conquest of England in 1066. In the segment depicted at the online site

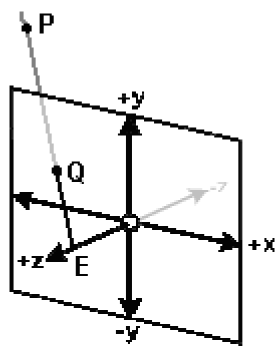
http://hastings1066.com/bayeux23.shtml

there are men eating at a table, and it looks as if the objects on the table are directly facing the viewer and ready to fall off the table's surface. In this and other segments of the tapestry one can also notice the flat appearance of nearly all objects. None of this detracts from the artistic value of the tapestry and some of the distortion can be explained because this is a tapestry rather than a painting, but medieval paintings also have many of the same traits. The following link contains a painting with a similar example involving tables whose tops appear as if they might be vertical.

http://www.mcm.edu/academic/galileo/ars/arshtml/renart1.html

The paintings of Giotto (Ambrogio Bondone, 1267 – 1337), especially when compared to those of his predecessor Giovanni Cimabue (originally Cenni di Pepo: 1240 – 1302), show the growing interest in visual accuracy quite convincingly, and other painters from the 14th and early 15th century followed this trend. It is interesting to look at these paintings and see how the artists succeeded in showing things accurately much of the time but were far from perfect. Eventually artists with particularly good backgrounds in Euclidean geometry began a systematic study of the whole subject and developed a mathematically precise theory of perspective drawing.

The basic idea behind the theory of perspective is illustrated below. Assume that the eye **E** is at some point on the positive **z**-axis, the canvas is the **xy**-plane, and the object **P** to be included in the painting is at the point **P** which is on the opposite side of the **xy** – plane as the eye **E.** Then the image point **Q** on the canvas will be the point where the line **EP** meets the **xy** – plane.



A detailed analysis of this geometrically defined mapping yields numerous facts that are logical consequences of the construction and Euclidean geometry. For example, one immediately has the following conclusion.

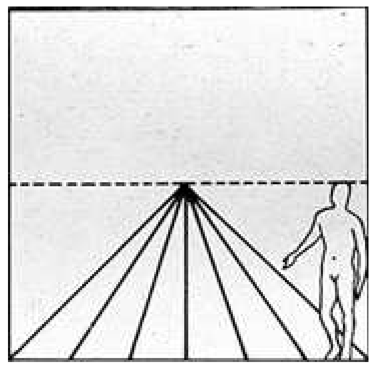
PROPOSITION. *If* **P**, **P′** *and* **P″** *are collinear points on the opposite side of* **E** *and* **Q**, **Q′** *and* **Q″** *are their perspective images, then* **Q**, **Q′** *and* **Q″** *are also collinear.*

PROOF. Let **L** be the line containing the three points. Then there is a plane **A** containing **L** and the point **E**; the three points **Q**, **Q′** and **Q″** all lie on the intersection of **A** with the **xy** – plane. Since the intersection of two planes is a line it follows that the three points must lie on this line.

Further analysis yields the following important observation.

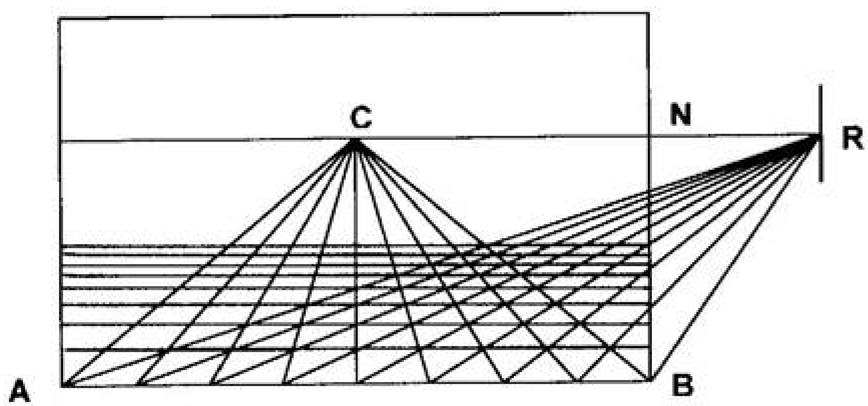
VANISHING POINT PROPERTY. *If* **L**, **M** *and* **N** *are mutually parallel lines then their perspective images pass through a single point on the* **xy** *– plane. This point is known as the vanishing point. The set of all vamishing points on all lines is the* **x –** *axis.*

Here is one picture to illustrate the Vanishing Point Property:



(***Source:*** http://www.collegeahuntsic.qc.ca/Pagesdept/Hist_geo/Atelier/Parcours/Moderne/perspective.html)

And here is another depicting two different families of mutually parallel lines:



(Source: http://www.math.nus.edu.sg/aslaksen/projects/perspective/alberti.htm)

It is enlightening to examine some paintings from the 14th and 15th centuries to see how well they conform to the rules for perspective drawing. In particular, the site

http://www.ski.org/CWTyler_lab/CWTyler/Art%20Investigations/Perspective History/Perspective.BriefHistory.html

analyzes Giotto's painting *Jesus Before the Caïf* (1305) and shows that the rules of perspective are followed very accurately in some parts of the painting but less accurately in others.

The first artist to investigate the geometric theory of perspective was Fillipo Brunelleschi (1377 – 1446), and the first text on the theory was *Della Pittura*, which was written by Leon Battista Alberti (1404 – 1472). The influence of geometric perspective theory on paintings during the fifteenth century is obvious upon examining works of that period. The most mathematical of all the works on perspective written by the Italian Renaissance artists in the middle of the 15^{th} century was *On perspective for painting (De prospectiva pingendi)* by Piero della Francesca (1412 – 1492). Not surprisingly there were many further books written on the subject at the time, of which we shall only mention the *Treatise on Mensuration with the Compass and Ruler in Lines, Planes, and Whole Bodies*, which was written by Albrecht Dürer (1471 – 1528) in 1525.

Here are some additional online references for the theory of perspective along with a few examples:

http://mathforum.org/sum95/math_and/perspective/perspect.html

http://www.math.utah.edu/~treiberg/Perspect/Perspect.htm

http://www.dartmouth.edu/~matc/math5.geometry/unit11/unit11.html

Here is a link to a perspective graphic that is animated:

http://gaetan.bugeaud.free.fr/pcent.htm

Of course, one can also use the theory of perspective to determine precisely how much smaller the image of an object becomes as it recedes from the **xy** – plane, and more generally one can use algebraic and geometric methods to obtain fairly complete quantitative information about the perspective image of an object. Such questions can be answered very systematically and efficiently using computers, and most of the time (if no always) the 3 – dimensional graphic images on computer screens are essentially determined by applying the rules for perspective drawing explicitly.

The consolidation of trigonometry

Although there is no specific date when the Middle Ages ended and the Renaissance began, the transition is generally marked by three events during the second half of the fifteenth century:

1. The invention of the printing press by Gutenberg in 1452.
2. The end of the Byzantine Empire with the Turkish conquest of Constantinople in 1453.
3. The (re)discovery of America by Columbus in 1492.

One could also add the end of the conquest of Granada and expulsion of the Moors from Spain in 1492, and for our purposes this is particularly significant because of the Arabic/Islamic influences on the history of mathematics. The level of mathematical activity in such cultures had been declining significantly ever since the12^{th} century.

Although there were still a few noteworthy contributors during the 15th century, there were none afterwards.

Following the Turkish conquest of Constantinople, many Greek scholars brought manuscripts of ancient Greek writers to Western Europe. These manuscripts led to more accurate and informed translations in many cases. Although the impact of the printing press for mathematics was not so immediate, it did lead to greatly increased communications among scholars and eventually to wider circulation of new ideas in mathematics.

New translations played a role in one significant mathematical development during the second half of the fifteenth century; namely, the emergence of trigonometry as a subject in its own right. Ever since Hellenistic times, trigonometry had been regarded by Greek, Indian and Arabic scientists mainly as a mathematical adjunct to observational astronomy. However, as trigonometry found increasingly many applications to other subjects such as navigation, surveying, and military engineering, it became clear that the subject could no longer be viewed in this fashion. The separation of these subjects was made very explicit in the work of Johann Müller of Königsberg (1436 – 1476), who is better known as ***Regiomontanus,*** which is a literal Latin translation of Königsberg (a city now called Kaliningrad that lies on the Baltic Sea in a small enclave of Russian territory between Poland and Lithuania). With his extremely broad interests and abilities, he was a perfect example of a Renaissance man. Regiomontanus made new translations of various classical works, and in his book *De Triangulis (On Triangles)* he organized virtually everything that was known in plane and spherical trigonometry at the time, from the classical Greek and Arabic results to more recent discoveries. In particular, this work systematically develops topics such as the determination of all measurements of a triangle from the usual sorts of partial data (side – angle – side *etc.*) and states the Law of Sines explicitly. In another work, *Tabulæ directionum*, he gives extensive trigonometric tables and introduces the tangent function. To provide an idea of the accuracy of his results, we note that his computations essentially give 57.29796 as the tangent of 89° and the correct value is 57.28896.

Fifty years ago subjects like solid geometry and spherical trigonometry were standard parts of the high school mathematics curriculum, but since this is no longer the case we shall include some online background references for spherical geometry and basic spherical trigonometry here:

http://www.math.uncc.edu/~droyster/math3181/notes/hyprgeom/node5.html

http://mathworld.wolfram.com/SphericalTrigonometry.html

http://star-www.st-and.ac.uk/~fv/webnotes/chapter2.htm

New directions in scientific thought

Not surprisingly, the rediscovery of ancient learning during the late Middle Ages and the revival of intellectual activity led to questions about how it should be carried forward. On one side there was interest in using the work of the ancient Greeks to study religious and philosophical issues, and on another side there was interest in putting this knowledge to practical use. Eventually both of these viewpoints found a place in late medieval and Renaissance learning, but the balance was weighted more towards the practical side than it had been in Greek culture. One clear manifestation of this in the sciences was the emphasis on systematic experimentation and finding clear, relatively simple explanations for natural phenomena. Mathematical knowledge during the late Middle Ages and Renaissance expanded in response to these increased practical and scientific needs.

Advances in mathematical notation

We have already noted the introduction of the Hindu-Arabic numeration system and some progress towards creating more concise ways of putting mathematical material into written form. Although some abbreviations and symbols had been introduced, only a few abbreviations of Italian words (like *cos* for *cosa* or unknown) had come anywhere close to being standard notation. However, during the 15th century mathematicians had begun to devise some of the symbols that we use today. Here is a short list of examples beyond those already mentioned.

Symbolism	***Year***	***Developer***
Juxtaposition for multiplication	15th C.	al-Qalasadi (1412 – 86)
	1544	M. Stifel
Superscripts for exponents	1484	N. Chuquet
+ and –	1489	J. Widman
√ (radical sign)	1525	C. Rudolff
=	1557	R. Recorde

Several other standard symbols date back to the seventeenth century and will be mentioned at that point in the notes.

A few additional comments about Chuquet (1445 – 1488) and his notational innovations deserve to be added. Namely, he introduced symbolism for the 0th power and also allowed the use of negative numbers as exponents.

The cubic and quartic formulas

We have already noted that accurate versions of the quadratic formula were known to the Babylonians by 2000 B. C. E., and that various mathematicians had devised methods for finding solutions cubic equations. However, for third and higher degree equations there was nothing comparable to the quadratic formula for finding solutions, and even at the end of the 15th century there were strong doubts that such formulas existed. During the early and middle 16th centuries mathematicians discovered formulas for the roots of cubic and quartic (fourth degree) polynomials in terms of the polynomial's coefficients. The colorful details of the discovery and publication of the cubic formula are recounted in Section 7.2 of Burton, particularly on pages 293 – 297. The mathematical aspects can be summarized by noting that the original formula was discovered in one basic case but not published by S. del Ferro (1465 – 1526), rediscovered independently and extended to other cases by Nicolo Fontana, who is better known as Tartaglia (1500

– 1557), and published without permission from Tartaglia by G. Cardano (1501 – 1576). As noted in Burton, although Cardano (or Cardan) had a well deserved reputation for being very unscrupulous, he also made important contributions to mathematics. His most important book, *Ars Magna*, was devoted to algebra as it was known at the time and included a great deal of important material that was unquestionably his own.

The main idea of the derivation of the cubic formula is to make a clever change of variables which takes a cubic equation in some variable **x** and transforms it into a quadratic equation in some related variable $\mathbf{z^3}$. One then solves for $\mathbf{z^3}$ by the quadratic formula and substitutes back to obtain the desired formula for **x**. We shall give a derivation of the basic formula using a simplification introduced by F. Viète (or Vieta), whose main work will be discussed later.

First of all, since the coefficient of $\mathbf{x^3}$ is nonzero and multiplying a polynomial by a nonzero constant does not change the roots, we may as well assume that the coefficient of this leading term is equal to 1. Thus we have an equation of the form

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0$$

and if we make the change of variables

$$z \equiv x - \tfrac{1}{3} a_2 .$$

we eliminate the quadratic term and obtain

$$x^3 + px = q.$$

The next step is to make the change of variables

$$x = w - \frac{p}{3w},$$

which leads to the equation

$$w^3 - \frac{p^3}{27w^3} - q = 0,$$

and if we clear this of fractions we obtain the following equation:

$$(w^3)^2 - q(w^3) - \tfrac{1}{27} p^3 = 0$$

We can solve this for $\mathbf{w^3}$ using the quadratic fomula and extract cube roots to find w itself. After doing this we may use the resulting values for **w** to find **x** and **z** in succession. The Cardan form of the solution is expressed in terms of **x** as follows:

$$x = \sqrt[3]{\sqrt{(p/3)^3 + (p/2)^2} + q/2} - \sqrt[3]{\sqrt{(p/3)^2 + (q/2)^2} - q/2}.$$

For the sake of completeness here is the explicit formula for the general cubic equation **$a\,x^3 + b\,x^2 + c\,x + d =$** 0 in terms of the coefficients:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

Clearly this formula is much more complicated than the quadratic formula, and in contrast to the latter it is not particularly useful for computing the roots of an arbitrary cubic polynomial. In fact, as Cardan noted, if one considers the equation

$$x^3 = 15x + 4$$

for which 4 is easily checked to be a root (and the remaining two roots are real), then the formula yields the initially surprising expression

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

There are many cubic polynomials for which one obtains such expressions as roots even in cases where all three roots are real; such cases were said to be ***irreducible***. Cardan did not know how to interpret the formula for the irreducible polynomial considered above, but subsequently R. Bombelli showed how to do so using complex numbers. In fact, Cardan mentioned complex numbers once in his most important book, *Ars Magna*, which was devoted to algebra as it was known at the time. In particular, Bombelli showed why the expression on the right hand side of the formula is equal to 4. This insight was an important step towards the ultimate acceptance of complex numbers by mathematicians by the beginning of the 19th century. Further details on these points appear in Section 7.3 of Burton, which also discusses the analogous formula for fourth degree equations due to L. Ferrari (1522 – 1565).

Although the cubic formula is generally not all that helpful for finding roots of polynomials and numerical methods are often indispensable for describing such roots, the cubic formula still ranks as one of the most important discoveries in mathematics because of its influence on mathematical thinking. First, of all it was a breakthrough that went far beyond anything that ancient mathematicians had done and showed that mathematics was poised to answer new sorts of questions. Also, the solution emphasized the need to work with negative numbers even complex numbers even in some problems that only seem to involve positive real numbers. As time progressed, numerous other examples of this sort arose. Finally, since the cosine of **3x** is a cubic polynomial in **cos x,** the cubic formula led mathematicians to view the classical Greek trisection problem in terms of algebra, and the success in finding a formula for third and fourth degree equations led to extensive studies of fifth degree equations during the next 250 years.

Late in the 18th century P. Ruffini (1765 – 1802) described a method for showing that one could not find a quintic formula; *i.e.*, an expression giving the roots of a general fifth degree equation in one variable in terms of addition, subtraction, multiplication and division and extraction of $\mathbf{n}^{\mathbf{th}}$ roots, where $\mathbf{n} =$ 2, 3, 4 or 5. Several years later, N. H. Abel (1802 – 1829) gave a more insightful and rigorous argument, and the same ideas show that there also cannot be a similar sort of formula for $\mathbf{n}^{\mathbf{th}}$ degree equations for any larger values of **n**. Actually it is possible to write down a quintic (fifth degree) formula if one introduces just one more operation; namely evaluation of a number using a function $\mathbf{g(x)}$ that is inverse to the polynomial $\mathbf{p(x) = x^5 + x}$; one can check that the function $\mathbf{p(x)}$ defines a 1 – 1 correspondence of the real line with itself because its derivative is always positive (hence it is strictly increasing) and the limits of $\mathbf{p(x)}$ as $\mathbf{x}$ tends to $\pm\infty$ are $\pm\infty$, and therefore an inverse function satisfying actually $\mathbf{x = g(x)^5 + g(x)}$ exists..

Here are some online references for quintic equations and some of the material discussed above. The second is an electronic version of a large poster covering nearly the entire history of research on such equations.

http://mathworld.wolfram.com/QuinticEquation.html

http://library.wolfram.com/examples/quintic/

The emergence of symbolic and decimal notation

By the middle of the 16th century various mathematical symbols and abbreviations were widely used, but some authors – for example, Cardano – still formulated much of their work rhetorically. However, towards the end of the century the situation changed rapidly, and by the beginning of the 17th century early versions of modern symbolic notation had become fairly well established.

The most influential contributor to the new symbolic notation was F. Viète (1540 – 1603), whom we have already mentioned in connection with the cubic formula. He brought together many scattered ideas that were in circulation and added some important ones of his own in his work, An *Introduction to the Art of Analysis (In artem analyticam isagoge)*. Here are some particularly significant new ideas he contributed.

1. He used letters in equations to denote both known and unknown quantities.
2. He consolidated different types of equations by using both addition and subtraction (thus allowing unified approaches to families of equations like

$x^2 = bx + c$, $x^2 + c = bx$ and $x^2 + bx = c$ which had been treated separately since the time of Al-Khwarizimi).

3. His use of symbolism was not casual but systematic.

Very similar ideas were advanced by T, Harriot (1560 – 1621), whose findings were not published until after his death. There are several unanswered questions about the extent to which Harriot and Viète were acquainted with each other's work or influenced each other.

Further discussion of Viète's work and R. Descartes' notational adjustments to it appear on 321 – 324 of Burton. Perhaps the most noteworthy thing to repeat here is that Viète used vowels to indicate unknowns and consonants to indicate known quantities, while Descartes used letters at the end of the alphabet to indicate unknowns and letters at the beginning of the alphabet to indicate known quantities.

Viète's other mathematical work included results on the theory of (roots of polynomial) equations, including the basic relationships between the coefficients of a polynomial and its roots. He also made highly significant contributions to trigonometry and the relationship between trigonometric identities and finding roots of polynomials; for example, his results yield solutions to the irreducible cubic equations that Cardano had attempted to understand, he gave a solution of the general cubic that requires the extraction of only a single cube root, and he showed that both the classical angle trisection and cube duplication problems depend upon solutions to cubic equations. The online site

http://math.berkeley.edu/~robin/Viete/construction.html

describes his construction of a regular 7-sided polygon using roots of cubic polynomials.

The following anecdote regarding Viète illustrates his use of trigonometry to find roots of polynomials. In 1593 A. van Roomen (1561 – 1615) issued an open challenge to solve an equation of the 45th degree

$$x^{45} - 45\,x^{43} + 945\,x^{41} + \ldots - 3795\,x^3 + 45\,x = K.$$

Viète noticed that the equation arises when one expresses $K = \sin 45\,y$ in terms of $x = 2\sin y$, and he quickly found about 20 roots to this equation (one reason he did not find more is that he only considered positive roots – complete and universal acceptance of negative numbers would finally take place by the middle of the 17th century, about 1000 years after Brahmagupta used them freely).

The end of the 16th century also saw the adoption of the decimal system essentially as we know it. In the discussion of Fibonacci (at least in Burton) it is noted that he still used Babylonian sexagesimal notation for fractions even though he used Hindu-Arabic numerals for whole numbers, and in fact for some time this was common practice. Some Arabic/Islamic mathematicians had discussed decimal fractions at length, most notably Jamshid al-Kashi (1380 - 1450) and Chinese mathematicians had also used decimals, but it was not until the appearance of *La Theinde* (also known as *Disme*) by Simon Stevin (1548 – 1620) that the use of decimal fractions became widespread in Europe (see pages 324 – 325 of Burton for additional information).