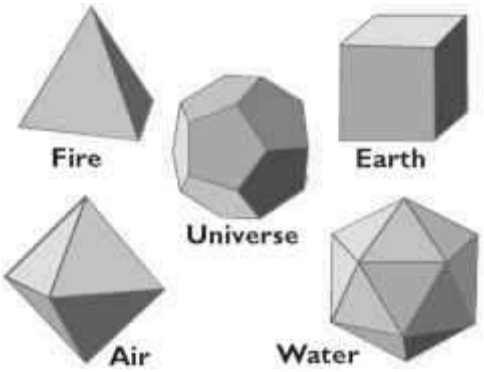
8.B. The result of Regiomontanus on tetrahedra

We have already mentioned that Plato's theory that the five regular polyhedra represent the fundamental elements of nature, and in Section 3**.**D of the notes we included the following graphical summary of his ideas:



However, Plato's student Aristotle disagreed with this, and his reasons were based upon geometrical properties of the regular polyhedra. The first step is a hypothesis on atomic and crystalline structure; namely, if pure forms of the elements existed then they should fill a region of space with the corresponding solid regular polyhedra in a manner analogous to the standard filling of space by solid cubes that only meet in common boundary faces or edges; for example, in the 3 – dimensional coordinate plane we can take the solid cubes whose coordinates satisfy

$$\mathbf{a \le x \le a + 1, \quad b \le x \le b + 1, \quad c \le x \le c + 1}$$

for suitable integers **a, b, c.** Aristotle claimed that the only solid regular polyhedra which fill space in a regular manner are the cube and the regular tetrahedron, but he did not give reasons for his assertion. Between the time of Aristotle and the late 15th century, there were many attempts to understand and prove his assertion about filling space with solid regular tetrahedra, but in the second half of the 15th century the previously mentioned scholar Regiomontanus proved that one cannot find a regular decomposition of 3 – dimensional space into solid regular tetrahedra that all have the same size. The main goal of this discussion is to explain why such decompositions cannot exist.

The result of Regiomontanus is historically significant because it marks a major advance in the recognition of flaws in the classical Graeco – Roman scientific theories. Some earlier work of this sort on mechanics had been done beginning in late 13th century, largely by scholars at Merton College in Oxford, and of course the 16th and early 17th centuries saw many revolutionary insights, of which the breakthroughs due to N. Copernicus and Galileo are probably the best known. Several topics in Regiomontanus' work anticipate the radical changes in scientific thought which took place during the next century. We have noted that his writings clearly mark the emergence of trigonometry as a subject in its own right, and trigonometric calculations play a key role in any disproof of Aristotle's assertion.

The following reference contains some historical background and further commentary on the questions considered here:

M. Senechal. ***Which Tetrahedra Fill Space?*** Mathematics Magazine, Vol. 54 (1981), pp. 227-243.

We shall first state the problem in precise mathematical terms, and we shall next describe some input from spherical geometry that will be useful. The main result will then follow fairly directly.

Statement of the problem

We shall formulate nearly everything in terms of solid analytic geometry and a preferred coordinate system for Euclidean 3 – space, and we shall denote geometric points by 3 – dimensional vectors whenever it is convenient to do so.

How does one define a regular solid tetrahedron analytically? The simplest characterization is in terms of the vertices. The four points w, x, y, z must satisfy the following conditions:

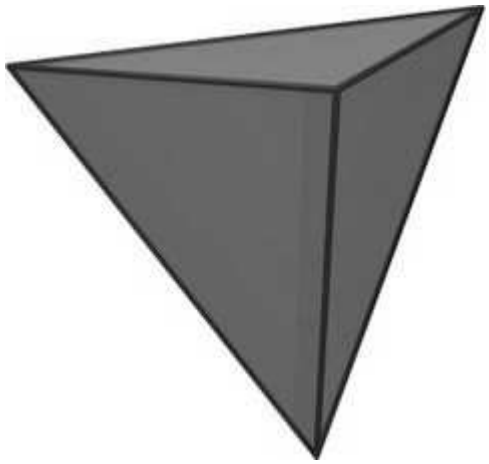
(a) No plane contains all four of them.

(b) The distances between all pairs of vertices are equal.

Of course, the second condition is analogous to the standard property for vertices of an equilateral triangle. ***The solid regular tetrahedron*** with vertices **w, x, y, z** is given by the set of all points **v** which can be written as linear combinations

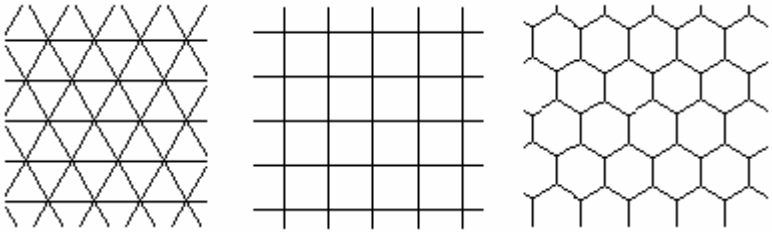
$$\mathbf{v} = \mathbf{aw + bx + cy + dz}$$

where $\mathbf{a, b, c, d \geq 0}$ and $\mathbf{a + b + c + d = 1}$. The four 2 – dimensional triangular ***faces*** of this tetrahedron are the solid triangles consisting of all points where one of the coefficients **a, b, c, d** is zero, and the six 1 – dimensional ***edges*** are the closed line segments consisting of all points where two of these coefficients are zero. The picture below depicts a solid regular tetrahedron; three of the four triangular faces are visible, and the remaining triangular face (given by the "outside" vertices) is hidden from view.



(Source: http://en.wikipedia.org/wiki/Tetrahedron)

What is a regular decomposition? We should expect this to resemble either the standard regular decomposition of 3 – space into cubes whose edges all have length one or any of the standard regular tilings (or tessellations) of the plane by equilateral triangles, squares or regular hexagons that are illustrated below:



Specifically, there should be a family of solid regular tetrahedra $\mathbf{A_p}$ such that all the following hold:

(a) The lengths of all the edges of all the (solid) tetrahedra $\mathbf{A_p}$ are equal.

(b) Every point of 3 – space lies in at least one of the tetrahedra $\mathbf{A_p}$**.**

(c) Two tetrahedra $\mathbf{A_p}$ and $\mathbf{A_q}$ meet in a common face, edge or vertex.

(d) The number of tetrahedra containing a vertex is the same for all vertices.

(e) The number of tetrahedra containing an edge is the same for all edges.

(f) The number of tetrahedra containing a face is equal to 2.

The first property says that all the regular tetrahedra have the same size (they automatically have the same shape), the second two say that one has a reasonable decomposition of space into such tetrahedral, and the final three are assumptions that the decomposition is uniform throughout space, not changing if one moves from one point or vertex or face to another. One can actually get by with a much shorter list of underlying assumptions, but then the mathematical proofs will be longer because each of our six conditions must eventually be verified.

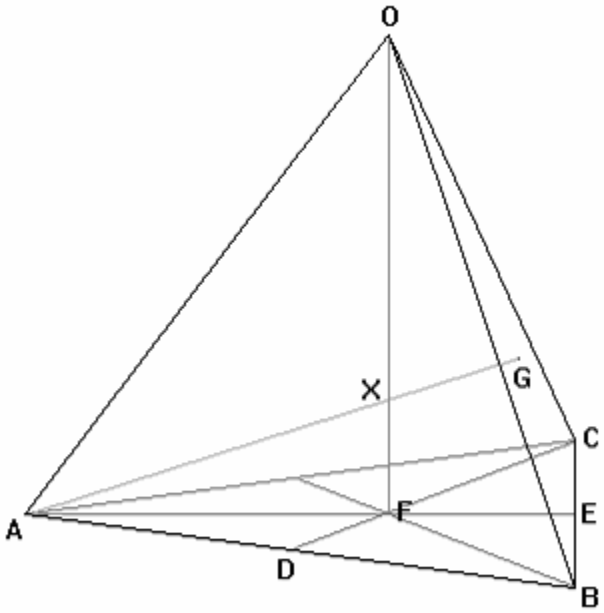
We shall need the following fact about regular tetrahedra.

Theorem. *Suppose that* **T** *is a regular solid tetrahedron in Euclidean* 3 *– space whose edges all have length equal to* **e,** *let* **O** *be a vertex of* **T,** *and let* **F** *be the center point of the face opposite* **O.** *Then the distance from* **O** *to* **F** *is equal to* **e · SQRT(**6**)**/3**.**

A derivation of this result in the special case **e = 1** appears at the online site

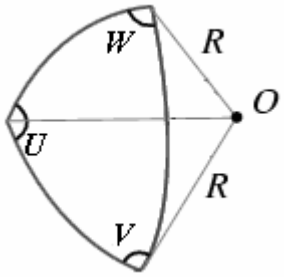
http://www.uwgb.edu/DutchS/symmetry/TetrahedronProps.htm

and a picture accompanying that derivation is reproduced on the next page for the sake of convenience. The coefficient **SQRT(**6**)**/3 is approximately equal to 0.8164966. Since **OF** is perpendicular to plane **ABC** and the shortest distance from **O** to the plane of **ABC** is along the perpendicular, it follows that ***if* X *is a point in the solid tetrahedron* OABC *such that the distance from* O *to* X *is strictly less than the quantity* e · SQRT(**6**)**/3, *then* X *does not lie on the face* ABC.**



Input from spherical geometry

Suppose now that we take the points which lie on the solid tetrahedron **T** and also on the sphere **S(R)** of radius **R** with center at the point **O,** where **R** is less than **e·SQRT(**6**)**/3**.** This intersection is a "closed spherical triangle" whose vertices **U, V, W** are the points where the segments **[OA], [OB], [OC]** meet the sphere **S(R).** An illustration is given below.



This picture is modified from an illustration at the following site:

http://mathworld.wolfram.com/SphericalTrigonometry.html

The lengths of the arcs joining **U** to **V, V** to **W,** and **W** to **U** all have equal length, and in fact they correspond to 60 degree angles. However, the measurements of the spherical vertex angles for the spherical triangle at **U, V** and **W** are **NOT** equal to 60 degrees; the measurements of these vertex angles are equal, but the value is strictly greater than 60 degrees (the sum of the vertex angle measurements for a spherical triangle is always greater than 180 degrees).

Computing the measurement of the vertex angles. For the sake of definiteness, let us consider the measurement of the angle at the vertex **V.** By Formula (31) in the

previously cited link http://mathworld.wolfram.com/SphericalTrigonometry.html we have the following general relationship connecting the measure of the vertex angle at **V** and the angular measurements **u, v, w** of the sides of the equilateral spherical triangle with vertices **U, V** and **W.**

$$\cos^2 (V/2) \quad = \quad \frac{\sin s \sin (s-u)}{\sin v \sin w}$$

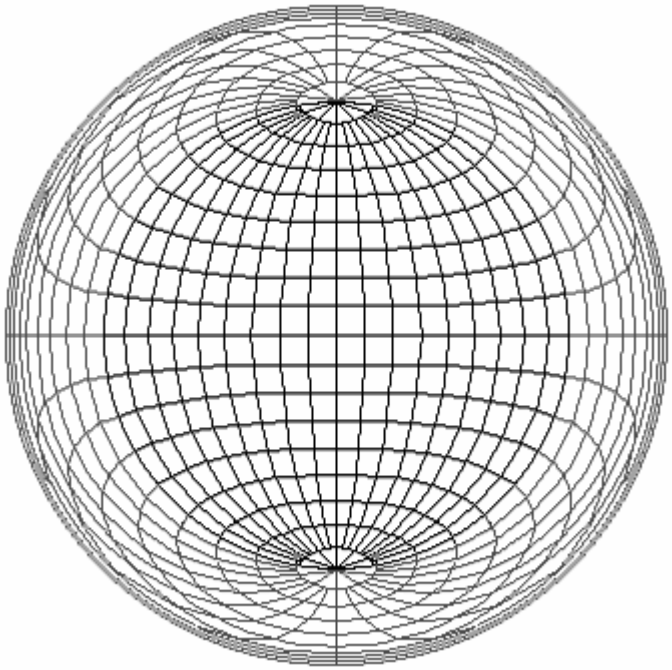
As in the classical Heron's Formula for the area of a triangle in terms of the lengths of its sides, we have **s = ½ (u + v + w).** We already know that **u = v = w =** 60 degrees, so it follows that the displayed expression is equal to 2/3. If we combine this with the basic formula

$$\cos 2\theta \quad = \quad 2\cos^2 \theta - 1$$

we find that the cosine of the vertex angle at **V** (hence also at **U** and **W**) is equal to 1/3, so that the degree angular measurement of the vertex angles in the spherical triangle are equal to approximately 70.528779 degrees because the latter angle is approximately equal to the inverse cosine of 1/3.

Conclusion of the argument

Suppose that we have a regular decomposition of space into regular tetrahedra as described previously, and let m be the number of tetrahedra containing a given edge. If we now look at the corresponding decomposition of the sphere **S(R)** into equilateral spherical triangles, then we see that each vertex **V** of the sphere decomposition also lies on exactly m spherical triangles. Furthermore, any two such triangles intersect in a common edge, and therefore the common value for the degree measurement of the vertex angles at **V** must be equal to 360/**m** degrees. If **V** is the North Pole or South Pole, this is particularly apparent from the illustration below:



(Source: http://www.uwgb.edu/dutchs/structge/sphproj.htm)

If we now let **m** range over the positive integers 1, 2, 3, **…** then the values obtained for 360/**m** are equal to 360, 180, 120, 90, 72, 60, and other positive values all of which are strictly less than 60. In particular, we see that the cosine of 360/**m** degrees is never equal to 1/3 when m Is a positive integer, contradicting the conclusions in the previous paragraph. Therefore it follows that there is no regular decomposition of 3 – dimensional space into regular tetrahedra.

Remarks on the other cases

Although it may seem intuitively clear that one cannot have regular decompositions of 3 – dimensional space into congruent regular octahedra, icosahedra or dodecahedra, a mathematical proof is necessary to show that no such decompositions can exist. A proof using more advanced methods is given in Section 4.6 of the following classic book:

Coxeter, H. S. M. Regular polytopes. Third edition. *Dover Publications, Inc., New York,* 1973. **ISBN-10:** 0486614808