11. Precalculus mathematics in the seventeenth century

(Burton, **8.1 – 8.2**)

During the past few decades the term precalculus has become a fairly standard term to describe several loosely related topics that are studied before calculus. These include some topics from algebra and coordinate (analytic) geometry, the basics of plane trigonometry, and standard material on exponential and logarithmic functions. By the middle of the seventeenth century much of this material was close to its present form. We shall discuss each area separately.

Standardization of algebra

Viète had taken the crucial steps in creating systematic symbolic notation for algebra in the late 16th century. However, several improvements were made during the early 17th century, particularly by R. Descartes (1596 – 1650). Here are some specific examples:

1. The current notation $\mathbf{x}^{\mathbf{n}}$ for powers of numbers; Chuquet had used superscripts for exponents in the 15th century but his notation did not include the variable or constant **x** being exponentiated, and Viète used words to denote power operations. Strictly speaking Descartes only used this notation when powers were nonnegative integers, but by the end of the century mathematicians used the notation freely for arbitrary real values of **n**; two individuals whose usage promoted extension of usage were J. Wallis (1616 – 1703) and I. Newton (1643 – 1727).

2. A shift from Viète's convention – namely, vowels represent unknowns and consonants represent known quantities – to the usual modern practice of using letters near the end of the alphabet to denote unknowns and letters near the beginning of the alphabet for known quantities; whenever we see an equation like $\mathbf{a\,x = b}$ today we normally assume that we are supposed to solve for **x** in terms of **a** and **b**.

3. Introduction of the term "imaginary roots" for complex, but not real, solutions to polynomial equations; the reasoning behind the term was that one could imagine such expressions as roots even though they do not correspond to any real quantity. If one interprets "real" in a strict mathematical sense this statement is correct, but if one thinks of "real" in a less formal sense this view can be disputed on two grounds. First, one can ask how "real" a real number is, and perhaps even more important, complex numbers do arise naturally in many physical contexts. For example, the concept of impedance in an alternating current electrical circuit is naturally measured by complex numbers.

Others also made significant contributions during this period, particularly T. Harriot (1560 – 1621), whom we mentioned earlier, A. Girard (1595 – 1632), and W. Oughtred (1574 – 1660). Although Harriot made important advances in the theory of equations and notation simplification, his impact was diminished because his writings were first published several years after his death and they were not well edited. He introduced a dot (but not a raised one) to denote multiplication; there is some disagreement whether this was meant as a symbol of operation or merely separating terms, but evidence for its operational use is substantial (in any case Leibniz did use such terminology specifically as an operation). More important, the standard inequality symbols $<$ and $>$ first appear in the published version of Harriot's work, *Artis Analyticæ Praxis ad Æquationes Algebraicas Resolvendas* (The Analytical Arts Applied to Solving Algebraic Equations), which was published in 1631. Substantial evidence indicates that Harriot himself did not introduce these symbols and they were inserted by the editors who published his work posthumously. Girard's contributions include the modern terminology for the six basic trigonometric functions we use today and further clarification of the notion of fractional powers. Oughtred introduced the so-called St. Andrew's Cross $\times$ for multiplication, and he also introduced the frequently used double colon symbol (**::**) for proportions or analogies; both appear in his work *Clavis Mathematicæ*, composed about 1628 and published in 1631. The *obelus* (÷) was first used as a division symbol by Johann Rahn (1622 – 1676) in 1659.

Trigonometry

Viète's work on trigonometry advanced the subject well beyond the high level it had reached in the 15th century work of Regiomontanus, and in particular Viète uses all six of the standard functions explicitly. We have already mentioned that the modern terminology for such functions was introduced by Girard, and his writings also contain the basic formula for the area of a spherical triangle (which depends upon the excess of the sum of its vertex angle measurements over 180 degrees).

Logarithms

Trignometric identities such as

$$\cos a \cos b \quad = \quad \tfrac{1}{2}\,[\ \cos(a-b) \quad + \quad \cos(a+b)\ ]$$

which relate sums and products of trigonometric functions, are examples of a concept known as *prosthaphaeresis* (from the Greek: *prosthesi* – addition and *afairo* – subtraction). During the late sixteenth century many mathematicians and astronomers, for example Tycho Brahe (1546 – 1601), recognized the usefulness of such formulas for reducing the amount of effort needed to carry out their computations. Enormous amounts of time were needed to complete the required multiplications, and the use of trigonometric identities as above provided a means for transforming a relatively hard multiplicative calculation into a relatively easy additive calculation. The development of logarithms was essentially a formalization of this procedure that eliminated the need to use trigonometry as an intermediary.

Towards the end of the 16th century work on logarithms began in independent work of J. Napier (1550 – 1617) and J. Bürgi (1552 – 1632). However, Napier published his findings much earlier, and Bürgi did not do so until Napier's work had become known and accepted by the scientific community. Napier's definitions and methods differ greatly from those in use today; considerable information on them appears on pages 327 – 330 of Burton (see also page 336), so we shall merely state his definition here: If

$$N = 10^7(1 - 10^{-7})^L$$

then **L** is defined to be the ***Napier logarithm*** of **N** (the term *logarithm* is also due to Napier from the Greek: *logos* – ratio and *arithmos* – number). Note that **L** is a *decreasing* function of **N** while in the modern definition for logarithms the latter is an increasing function. These logarithms convert products to sums, for if

$$N_1 = 10^7(1 - 10^7)^{L_1} \qquad N_2 = 10^7(1 - 10^{-7})^{L_2}.$$

then we have

$$N_1 N_2 = 10^7[10^7(1 - 10^{-7})^{L_1 + L_2}].$$

Napier's work had an very strong immediate and near universal impact on mathematical computations.

The switch to common (base 10) logarithms came from discussions between Napier and H. Briggs (1561-1631). Briggs carried out the work needed to construct new tables and completed it seven years after Napier's death.

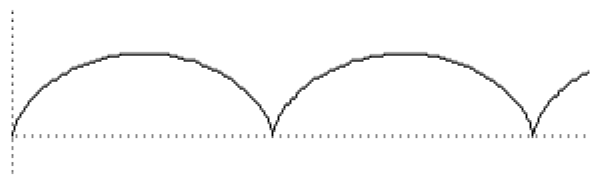
Coordinate (analytic) geometry

> As long as algebra and geometry proceeded along separate paths, their progress was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality, and thenceforward marched on at a rapid pace towards perfection.
>
> J. – L. Lagrange (1736 – 1813)

We have already noted that the representation of points by numerical coordinates had appeared in some earlier work, implicitly in the writings of Apollonius and more firmly in the work of Oresme in the 14th century. The latter's ideas circulated widely during the

16th century, and in particular they are present in the work of Galileo Galilei (1564 – 1642). We shall comment on some mathematical aspects of his work that illustrated the need for something like coordinate geometry. All of these are related to his monumental discovery that an object falling to the ground without resistance will do so with uniformly accelerated motion. This concept had been discussed since the late middle ages long before its practical significance was understood. One basic consequence of this was that the motion of a projectile fired at some upward angle is a parabola (neglecting air resistance), showing that these curves – which were originally introduced because they were interesting mathematically – had major practical applications.

Galileo was interested in a variety of questions concerning curves and their properties. One particular curve that attracted his interest was the cycloid describing the motion of a point on a rolling wheel, which had been introduced by Nicholas of Cusa (1401 – 1464).



Galileo was also interested in finding a curve that is isochronic curve; *i.e.*, a curve such that, from any point, the object dropped will take the same amount of time to reach the bottom. He thought such a curve was given by a segment of a circle, but somewhat ironically it turns out that the cycloid is an isochronic curve. A proof of this was given by C. Huyghens (1629 – 1695). Another illustration of the urgent need for coordinate geometry was Galileo's attempt to study the shape of a hanging chain, where he incorrectly concluded that it defined a parabola rather than a catenary (the graph of the hyperbolic cosine function). We shall conclude this discussion by repeating its purpose; namely, to point out that there was a serious need for better ways of analyzing geometric problems, particularly many that arose from physics.

The two most prominent names in the development of analytic geometry are R. Descartes and P. Fermat, who worked independently of each other although each of them eventually had at least some knowledge of the other's work. In commenting on his scientific legacy, Isaac Newton made the following well known statement in a letter to R. Hooke:

> If I have been able to see further, it was only because I stood on the shoulders of giants.

The same comment applies equally well to the work of both Descartes and Fermat. In both cases their work was based upon (1) familiarity with the insights of Viète on studying geometrical questions by algebraic methods, (2) a desire to understand or reconstruct the deep and partially lost results of Apollonius and others on conics and related figures. Fermat was a mathematician by avocation rather than profession, and virtually none of his work was published during his lifetime. In contrast, Descartes was a scholar by profession and published his findings promptly, and largely because of this he generally receives most of the credit for developing coordinate geometry. Aside from his publication of his findings and the rapid adoption of his ideas by those who read his works, Descarts' superior algebraic notation marks one more respect in which he surpassed Fermat. On the other hand, his writings on geometry were often vague, and many things often associated with his name – for example, orthogonal coordinate axes and explicit description of points as ordered pairs of numbers – are not mentioned at all in his work. Also absent are formulas for basic quantities like slope and distance. There is no full use of negative coordinates, and no new curves are plotted using coordinates. As Boyer says in his history of mathematics, "Descartes was probably the most able thinker of his day, but at heart he was not a mathematician." This view is underscored by the fact that his discussion of coordinate geometry is formally just part of one addendum, *La Géométrie,* to his work, *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Discourse on the Method of Correctly Reasoning and Seeking Truth in the Sciences), which is one of the most important and influential books on the philosophy of science that has ever been written.

One noteworthy advance in Descartes' work is his willingness to grant higher order algebraic plane curves the same legitimacy as lines and circles. However, he also distinguishes carefully between such curves and ***transcendental*** (mechanical) curves, the distinction being his view that ***algebraic*** (geometric) curves could be described exactly while mechanical curves, for example the classical Quadratrix of Hippias, are given by two separate movements. Subsequent progress in mathematics during the next century or so led mathematicians to view all such "mechanical" curves as legitimate, and in the nineteenth century mathematicians also realized the needed to work with still other classes of curves that could not even be constructed by any mechanical means.

To illustrate the applicability of his methods, Descartes uses them to give new derivations of some classical results in Greek geometry due to Apollonius and Pappus.

Although both Descartes and Fermat began with the same basic sources and were led to the same idea of specifying positions by means of what we now call coordinates, their emphases were often entirely different. One basic difference was that Descartes started with a curve and then derived an equation for it, while Fermat started with an equation and then described the curve. Fermat's approach led him directly to the standard first and second degree equations for lines and conics. He further uses his methods to reprove old and new geometrical results including the following one:

Theorem. *Given any number of fixed lines, the set of points such that the sum of the squares of the segments drawn at given right angles from the point to the lines is constant, is an ellipse.*

This illustrates the power of the methods, for such a result would be nearly impossible to prove without analytic geometry. However, with the help of algebra it is fairly simple, because the condition in the theorem can be expressed as

$$\sum_{n=1}^{N}\left[\frac{a_n x + b_n y + c_n}{\sqrt{a_n^2 + b_n^2}}\right]^2 = k.$$

which is the equation of an ellipse.

One can summarize the differences between Descartes' and Fermat's treatments of coordinate geometry by saying that Fermat's exposition and clarity were superior to Descartes', his analytic geometry is closer to our own, and in particular he uses rectangular coordinates just as we do today.

At various points we have noted that even during the 17^{th} century mathematicians were not always ready to use negative numbers just as freely as positive ones. There does not seem to be any clear point at which most mathematicians accepted (surely with reluctance in many cases) the use of negative numbers in algebraic formalism, but one important step was when, in 1657, J. Hudde (1633-1704) was first writer to let letters represent negative as well as positive numbers.

Fermat's work on number theory

Much of Fermat's mathematical legacy involves his work on number theory, so we shall mention just a few points related to topics raised earlier in these notes. In particular, we have already mentioned the Euclid – Euler characterization of even perfect numbers, which involves primes of the form $2^{\mathbf{n}} - 1$. Fermat is credited with showing that a number of this form can be prime only if **n** is prime. The proof is a striking illustration of symbolic manipulations:

$$\begin{aligned} 2^n - 1 &= 2^{rs} - 1 = (2^r)^s - 1 \\ &= (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} \cdots + 2^r + 1). \end{aligned}$$

Here are two further results along the same lines:

(1) *If* **p** *is odd prime, then* 2**p** *divides* $2^{\mathbf{p}} - 2$ *or equivalently* **p** *divides* $2^{\mathbf{p}-1} - 1$.

(2) *If* **p** *is an odd prime, then every divisor of* $2^{\mathbf{p}} - 1$ *has the form* 2 **p k** + 1.

The second result reduces the number of divisors one must check to determine if $2^{\mathbf{p}} - 1$ is prime. Fermat did not give proofs for his number theoretic results, but he often gave some examples. As noted in the MacTutor site's biographical sketch, "Everyone failed to see that Fermat had been hoping his specific problems would lead them to discover, as he had done, deeper theoretical results. "

A generalization of (1) is a standard item in undergraduate number theory, discrete structures and abstract algebra courses:

Fermat's Little Theorem. *If* **p** *is any prime and* **a** *is any positive integer, then* **p** *divides* $\mathbf{a}^{\mathbf{p}} - \mathbf{a}$ **.**

The first published proof is due Euler (1732) in a much more general form; Leibniz left an earlier proof in manuscript form.

In the earlier discussion of Diophantine problems (and the material on Fibonacci) it was noted that no numbers of the form $4k + 2$ or $4k + 3$ could be realized as sums of two squares. Conversely, one can ask which of the remaining numbers can be so represented. In his writings Fermat stated that every prime number of the form $4k + 1$ could be so represented; once again the first published proof was due to Euler.

Finally, no discussion of Fermat and number theory can be complete without mentioning **Fermat's Last Theorem**:

> The equation $\mathbf{x}^n + \mathbf{y}^n = \mathbf{z}^n$ *has no solutions in positive integers if* $\mathbf{n} > 2$.

It is beyond the scope of this course to recount the entire history of work on this problem or of its ultimate solution near the end of the 20^{th} century. However, we should note that the cases $\mathbf{n} = 3, 4$ can be disposed of by techniques developed in upper level undergraduate (or introductory graduate) level abstract algebra courses, and proofs for many other values of **n** were obtained over the years. During the 1980s a major breakthrough was made by G. Faltings (1954 –), who showed that there were at most finitely such solutions for which **x**, **y** and **z** were relatively prime. This result required an enormous amount of mathematical machinery that had been developed in the meantime. Fermat’s Last Theorem was finally established by A. Wiles (1953 –) in 1994, with contributions from R. Taylor; this proof also required a great deal of mathematics.

A natural question about Fermat’s Last Theorem concerns the value of the solution. The result by itself does not have any known “practical” applications of its own. However, the solution of the problem is significant because it illustrates the power of mathematical methods that were developed during the 350 years between Fermat’s statement of the result and its proof, and it is also significant because the problem directly or indirectly led to profound developments in many mathematical areas that have found broad ranges of uses both in mathematics itself and also in other subjects. As indicated above, Fermat would have been very pleased with this outcome.