12. The development of calculus

13. Newton and Leibniz

(Burton, **8.3 – 8.4**)

14. Calculus in the eighteenth century

(Burton, **9.3, 10.2**)

We have already noted that certain ancient Greek mathematicians – most notably Eudoxus and Archimedes – had successfully studied some of the basic problems and ideas from integral calculus, and we noted that their method of exhaustion was similar to the modern approach in some respects (successive geometric approximations using figures whose measurements were already known) and different in others (there was no limit concept, and instead there were delicate *reductio ad absurdum* arguments). Towards the end of the 16th century there was a revival of interest in the sorts of problems that Archimedes studied, and later in the 17th century this led to the independent development of calculus by Isaac Newton (1643 – 1727) and Gottfried Wilhelm von Leibniz (1646 – 1716).

Two important factors leading to the development of calculus originated in the late Middle Ages, and both represented attempts to move beyond the bounds of ancient Greek mathematics, philosophy and physics.

1. <u>*Interest in questions about infinite processes and objects.*</u> As noted earlier, ancient Greek mathematics was not equipped to deal effectively with Zeno's paradoxes about infinite and by the time of Aristotle it had essentially insulated itself from this "horror of the infinite." However, we have also noted that Indian mathematicians did not share this reluctance to work with the concept of infinity, and Chinese mathematicianss also used methods like infinite series when these were convenient. During the 14th century the mathematical work of Oresme and others on infinite series complemented the interests of the Scholastic philosophers; questions about the infinite played a key role in their efforts to make Greek philosophy consistent with Christianity. Philosophers such as William of Ockham (1285 – 1349) and Gregory of Rimini (1300 – 1358) provided insights which anticipated the concept of limit that is basic to calculus as we know it.

2. <u>*Interest in questions about physical motion.*</u> During the 13th century Jordanus discovered the mathematically correct description for the physics of an inclined plane (a result that

had eluded Archimedes and was described incorrectly by Pappus), and later members of the Merton school in Oxford such as T. Bradwardine (1290 – 1349) and W. Heytesbury (1313 – 1373) had discovered an important property of uniformly accelerated motion; namely, the ***average velocity*** (total distance divided by total time) is in fact the mathematical average of the initial and final velocities. The significance of this concept for the motion of falling bodies was not understood at the time and would not be known until the work of Galileo and others in the 16th century. During the late 16th century interaction between mathematics and physics began to increase at a much faster rate, and many important contributors to mathematics such as S. Stevin also made important contributions to physics (in Stevin's case, the centers of mass for certain objects and the statics and dynamics of fluids).

Here are some online references to further information about the topics from the Middle Ages described above:

http://www.math.tamu.edu/%7Edallen/masters/medieval/medieval.pdf

http://www.math.tamu.edu/%7Edallen/masters/infinity/infinity.pdf

http://plato.stanford.edu/entries/heytesbury/

Problems leading to the development of calculus

In the early 17th century mathematicians became interested in several types of problems, partly because these were motivated by advances in physics and partly because they were viewed as interesting in their own right.

1. Measurement questions such as lengths of curves, areas of planar regions and surfaces, and volumes of solid regions, and also finding the centers of mass for such objects.

2. Geometric and physical attributes of curves, such as tangents and normals to curves, the concept of curvature, and the relation of these to questions about velocity and acceleration.

3. Maximum and minimum principles; *e.g.*, the maximum height achieved by a projectile in motion or the greatest area enclosed by a rectangle with a given perimeter.

Specific examples of all three types of problems had already been studied by Greek mathematicians. The results of Eudoxus and Archimedes on the first type or problem and of Archimedes and Apollonius on the second have already been discussed. We know about the work of Zenodorus (200 – 140 B.C.E.) on problems in the third type because it is reproduced in Pappus' anthology of mathematical works (the *Collection* or *Synagoge*). Some examples of Zenodurus' results are

(1) among all regular n-gons with a fixed perimeter, a regular n-gon encloses the greatest area,

(2) given a polygon and a circle whose perimeter of the polygon equals the circumference of the circle, the latter encloses the greater area,

(3) a sphere encloses the greatest volume of all surfaces with a fixed surface area.

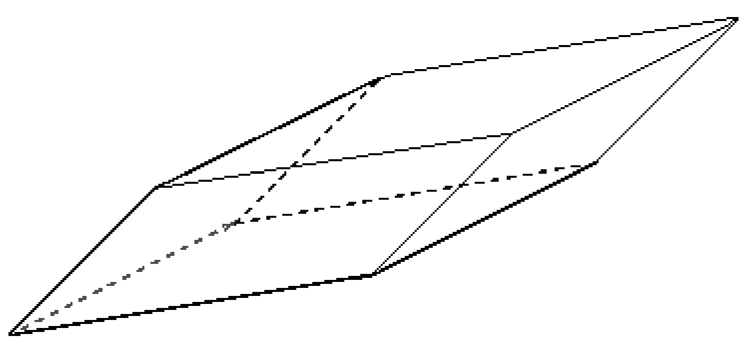
Infinitesimals and Cavalieri's Principle

We shall attempt to illustrate the concept of infinitesimals with an example of use to determine the volume of a geometrical figure.

Usually the formula for the volume of a cone is derived only for a ***right circular cone*** in which the axis line through the vertex or nappe is perpendicular to the plane of the base. It is naturally to ask whether there is a similar formula for the volume of an oblique cone for which the axis line is not perpendicular to the plane of the base. One can answer this question using an approach developed by B. Cavalieri (1598 – 1647). A similar idea had been considered by Zu Chongzhi, who is also known as Tsu Ch'ung-chin (429 – 501), but the latter's work was not known to Europeans at the time.

CAVALIERI'S PRINCIPLE. *Suppose that we have a pair three-dimensional solids* **S** *and* **T** *that lie between two parallel planes* $\mathbf{P}_1$ *and* $\mathbf{P}_2$, *and suppose further that for each plane* **Q** *that is parallel to the latter and between them the plane sections* $\mathbf{Q} \cap \mathbf{S}$ *and* $\mathbf{Q} \cap \mathbf{T}$ *have equal areas. Then the volumes of* **S** *and* **T** *are equal.*

Here is a physical demonstration which suggests this result. Take two identical decks of cards that are neatly stacked just as they come right out of the package. Leave one untouched, and for the second deck push along one of the vertical edges so that the deck forms a rectangular parallelpiped as below.



In this new configuration the second deck has the same volume as first and it is built out of very thin rectangular pieces (the individual cards) whose areas are the same as those of the corresponding cards in the first deck. So the areas of the plane sections given by the separate cards are the same and the volumes of the solids formed from the decks are also equal.

We shall now apply this principle to cones. Suppose we have an oblique cone as on the right hand side of the figure below. On the left hand side suppose we have a right circular cone with the same height and a circular base whose area is equal to that of the elliptical base for the second cone.

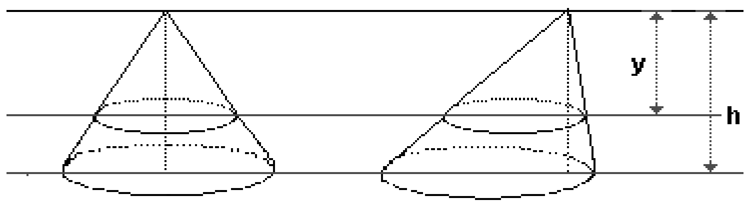


Figure illustrating Cavalieri's Principle

In the notation of Cavalieri's Principle we can take $\mathbf{P_1}$ to be the plane containing the bases of the two cones and $\mathbf{P_2}$ to be the plane which contains their vertices and is parallel to $\mathbf{P_1}$**.** Let **Q** be a plane that is parallel to both of them such that the constant distance between $\mathbf{P_2}$ and **Q** is equal to **y;** we shall let **h** denote the distance between $\mathbf{P_1}$ and $\mathbf{P_2}$**,** so that **h** is also the altitude of both cones. If **b** denotes the areas of the bases of both cones then the areas of the sections formed by intersecting these cones with **Q** are both equal to $\mathbf{b\ y^2/h^2}$**.** Therefore Cavalieri's Principle implies that both cones have the same volume, and since the volume of the right circular cone is $\mathbf{b\ h^2}/\ 3$ it follows that the same is true for the oblique cone.

One approach to phrasing our physical motivation in mathematical terms is to imagine each cone as being a union of a family of solid regions given by the plane sections; for the right circular cone these are regions bounded by circles, and for the oblique cone they are bounded by ellipses. Suppose we think of these sections as representing cylinders that are extremely thin. Then in each case one can imagine that the volume is formed by adding together the volumes of these cylinders, whose areas – and presumably thicknesses – are all the same, and of course this implies that the volumes of the original figures are the same. One fundamental question in this approach is to be more specific about the meaning of "extremely thin." Since a planar figure has no finite thickness, one might imagine that the thickness is something less, and this is how one is led to the concept of a thickness that is ***infinitesimally small.***

The preceding discussion suggests that an infinitesimal quantity is supposed to be nonzero but is in some sense smaller than any finite quantity. If we are given a geometric figure that can be viewed as a union of "indivisible" objects with one less dimension – for example, the planar region bounded by a rectangle viewed as a union of line segments parallel to two of the sides, or the solid region bounded by a cube as a union of planar regions bounded by squares parallel to two of the faces – then the idea is to view the square as a union of rectangular regions with infinitesimally small width or the square as a union of solid rectangular regions with infinitesimally small height. Likewise, this approach suggests an interpretation of a continuous curve as being composed of a family of straight lines with infinitesimally small length. When applied to our examples of cones, it leads to thinking of the solid region bounded by either cone as a union of circular or elliptical cylinders with infinitesimally small height.

Mathematicians and users of mathematics have though about infinitesimals for a long time. . They already appear in the mathematics of the early Greek atomist philosopher Democritus (460 – 370 B.C.E.) and an approach to squaring the circle developed by Antiphon, but advances by Eudoxus and others during the 4th century B.C.E. enabled Greek mathematicians to avoid the concept, and this fit perfectly with the reluctance of Greek mathematicians and philosophers (*e.g.*, Aristotle) to eliminate questions about the infinite from their mathematics. Taking the somewhat obscure form of "indivisibles", they reappeared in the mathematics of the late Middle Ages, and they played an important role in the work of J. Kepler (1571 – 1630) on laws of planetary motion, particularly his Second Law which states that the orbits of planets around the sun sweep out equal areas over equal times. During the 17th century infinitesimals were used freely by many mathematicians and scientists who contributed to the development of calculus, and in particular both Newton and Leibniz used the concept in their definitive accounts of the subject. However, as calculus continued to develop, doubts about the logical soundness of infinitesimals also began to mount. Such questions ultimately had very important consequences for the development of mathematics, and they will be discussed later in these notes.

A general comment on coverage

During the 17th century many mathematicians were interested in similar problems, and many results were discovered independently by two or more researchers. Not all such cases can be described completely in a brief summary such as these notes; one guiding principle here is to mention the persons whose work on a given problem had the most impact.

Progress on measurement questions

Methods and results from Archimedes and others provided important background and motivation for work in this area. We have already discussed the use of infinitesimals to derive formulas in some cases, and there was a great deal of further work based upon such ideas. In particular, during the time before Newton's and Leibniz' work appeared many of the standard examples in integral calculus had been worked out by preliminary versions of methods that became standard parts of the subject. Here are some specific examples:

Integrals of polynomials and more general power functions. Cavalieri computed the integral of $\mathbf{x^n}$ geometrically in cases where **n** is a positive integer, Gregory of St. Vincent (1584 – 1667) integrated $\boldsymbol{x^{-1}}$ in geometric terms that are equivalent to the usual formula of **ln x**, and J. Wallis (1616 – 1703) generalized the integral formula for $\mathbf{x^n}$ to other real values of **n**. Wallis was a particularly important figure in the development of calculus for several reasons. His methods, which are discussed on pages 357 – 360 of Burton, replace geometric techniques with algebraic computations and analytic considerations, and as such they are a milestone in the development of analysis (calculus) as a subject distinct from both algebra and geometry. In an entirely different direction, Wallis is also known today for his applications of integral formulas to derive his infinite product formula for $\pi/2$:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

We should note here that an earlier infinite product formula involving π

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

had been discovered by Viète (and, as noted earlier, Indian mathematicians had already discovered some of the standard infinite serieis formulas involving π).

Measurements involving the cycloid curve and regions partially bounded by it. There was an enormous amount of interest in the properties of the cycloid curve during the 17th century, and there were also some bitter disputes about priorities among some of the numerous mathematicians who worked on this example. Results included computations of arc lengths as well as areas, volumes and centers of mass associated to the curve. Mathematicians whose names are associated with this work include B. Pascal (1623 – 1662), E. Torricelli (1608 – 1647), and G. P. de Roberval (1602 – 1675).

Infinite solids with finite volume. Torricelli also discovered a fact that few if any mathematicians had anticipated; namely, the existence of an unbounded solid of revolution whose volume is finite; his example is an unbounded piece of the solid formed by rotating the standard hyperbola $\mathbf{y} = 1/\mathbf{x}$ about the **x –** axis.

Arc length. Early in the 17th century there were doubts about the possibility of computing the arc lengths of many curves, including some extremely familiar examples. Results of H. van Heuraet (1633 – 1660) showed the problem of finding arc length of a given curve is equivalent to determining the area under another curve, and he also worked out certain examples including the semi-cubical parabola $\mathbf{a\,y^2 = x^3}$**.** The arc length of a spiral curve was computed by Roberval.

Integrals and series expansions of transcendental functions. Results on the integrals of standard trigonometric functions were obtained by Pascal, Roberval, I. Barrow (1630 – 1677) and J. Gregory (1638 – 1675). The standard infinite series expansion for **arctan x** was obtained by Gregory (as noted before, the Indian mathematician Madhava had discovered two centuries earlier), and the standard infinite series expansion for **ln (1 + x)** was obtained by N, Mercator (1620 – 1687); the latter should not be confused with the mathematical cartographer G. Mercator (1512 – 1592) after whom a familiar type of map is named. Gregory also made numerous other contributions, including extending and applying the classical method of exhaustion to questions about other conic sections and writing the first text covering the material that would become calculus.

Progress in differentiation and maximization/minimization problems

Greek mathematics provided far less insight into questions about tangent lines or maximizing functions than it did for computing areas and volumes. In particular, there were no general principles comparable to the method of exhaustion for describing tangents; each example was treated in an entirely separate manner. Therefore one major problem facing 17^{th} century mathematicians was to produce a workable concept of tangent line.

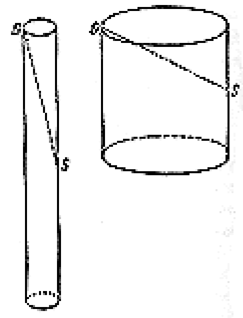
Some names particularly associated with this problem are Descartes, Fermat, Roberval and Barrow. Descartes' approach was based on finding the normal (perpendicular) line to a curve at a point, and Roberval's was motivated by the standard interpretation of a parametrized curve as the path of a moving object. Fermat and Barrow both defined tangent lines by the method that has become standard; namely, the tangent line to a curve **C** at a point **x** is a limit of the secant lines joining **x** to a second point **y** on **C** as **y** approaches **x.** Fermat had the basic idea, but Barrow's language (*i.e.*, his ***differential triangle***) was more precise. A discussion of Barrow's work appears on pages 363 – 364 of Burton.

As in the case of measurement problems, examples were the focal point of work on tangents. The standard results on the slopes of tangent lines for polynomial graphs were obtained by Fermat in the monomial case and in complete generality by Hudde. Applications of derivatives to repeated roots of polynomials were also discovered at this time. Tangent lines to the cycloid were determined independently by Descartes, Fermat and Roberval.

Fermat also studied maximization and minimization problems using the approach he developed for tangent lines. We note that he was interested in several different types of minimization problems in mathematics and physics, including the determination of a point inside a triangle such that the sum of the distances to all three sides is minimized, and more significantly his Least Time Principle in optics, which states that a beam of light will take the path from one point to another that takes the shortest amount of time and yields the standard results on refraction and reflection. Further discussion of such basic minimization problems is contained in the following book:

S. Hildebrandt and A. Tromba, ***The Parsimonious Universe***.
Springer – Verlag, New York, 1996. ISBN: 0 – 387 – 97991 – 3**.**

A result of Kepler's on maximization {the ***Wine Barrel Problem***) can also be mentioned at this point. He observed that a wine merchant had figured the amount of wine in a barrel by inserting a measuring stick into the tap hole **S** until it reached the lid at **D**, after which he read off the length **SD = d** and set the price accordingly. Kepler was concerned about the uniformity of pricing by this method and decided to analyze its accuracy; he correctly realized that a narrow, high barrel might have the same linear measure **SD** as a wide one and would indicate the same wine price, though its volume would be considerably smaller (see the figure on the next page).



In his approach to determine the volume in terms of **d** to determine the volume, Kepler approximated the barrel by a cylinder, with base radius **s** of the base and height **h.** He then looked for the value of **h** giving the largest value **V** if **d** is held constant. Using differential calculus one can show that the relation between **d** and **h** must be $3\mathbf{h}^2 = 4\mathbf{d}^2$ which is what Kepler found using less refined methods. He also observed that the shapes of the wine barrels were close to this optimal value – so close that he could not believe this was a coincidence. Of course, the manufacturing processes then were less uniform than they are now, so is was unlikely that all barrels satisfied this mathematical specification precisely, but Kepler further noted that if a barrel deviated slightly from the optimal ratio this would have little effect on the volume because a function changes very slowly near its maximum.

The emergence of calculus

Since many of the basic facts in calculus were known before the work of Newton and Leibniz, it is natural to ask why they are given credit for inventing the subject. Others came very close to doing so; in particular, Barrow understood that the process for finding tangents (differentiation in modern terminology) was inverse to the process for finding areas (integration in modern terminology). In these notes we shall focus on the decisive advances that make the work of Newton and Leibniz stand out from the important, high quality results due to many of their contemporaries.

1. Before Newton and Leibniz techniques for differentiating and integrating specific examples had been developed, but they were the first to set general notation and define general "algorithmic processes" for each construction. Earlier workers were not able to derive useful and general problem-solving methods.

2. Newton and Leibniz recognized the usefulness of differentiation and integration as general processes, not just as methods to solve measurement and tangent problems in important special cases. No had previously recognized the usefulness of calculus as a general mathematical tool.

3. With the exception of Barrow, the inverse relationship between differentiation and integration had not been clearly recognized

in earlier work, and Newton and Leibniz were the first to formulate it explicitly and establish it in a logically convincing manner.

4. Both stated the main ideas and results of calculus algebraically, so that the subject was no longer an offshoot of classical Greek geometry but significantly broader in scope and poised to make fundamental contributions to many areas of knowledge.

Sections 8.3 and 8.4 of Burton contain a great deal of detail about the scientific and philosophical contributions of Newton (see pages 365 – 381) and Leibniz (see pages 383 – 402). In particular, the bitter dispute about credit for discovering calculus is described there. We shall only summarize the points that are now generally accepted: The discoveries of Newton and Leibniz were essentially independent, and although Newton was the first to develop the subject, Leibniz published his version first. We should add that the discoveries by Newton and Leibniz took place around 1665 and 1673 respectively, Leibniz' work was published in 1684 while Newton's was published 1736, nearly a decade after his death.

Rather than dwell on the dispute over priorities, we shall discuss a few substantive similarities and differences between the work of Newton and Leibniz.

Many of the similarities were already mentioned in the reasons why Newton and Leibniz are given credit for creating calculus. One additional similarity is that each used both differentiation and integration to solve difficult and previously unsolved problems. Both also proved many of the same basic results; *e.g.*, the standard rules for differentiating functions, the Fundamental Theorem of Calculus, and the basic formal integration techniques which appear in calculus textbooks. On the other hand, Newton and Leibniz clearly had different priorities and these can be seen in the differences between their approaches and conclusions.

1. The fundamentally important binomial series expansion for $\mathbf{(1+x)^r}$, where $\mathbf{r}$ is an arbitrary real number and $|\mathbf{x}| < 1$, is solely due to Newton.

2. Newton used the words *fluxion* and *fluent* to denote the derivative and integral, and he denoted derivatives by placing dots over variables. Leibniz ultimately adopted the **dx** notation and the integral sign that are used today.

3. Newton was primarily interested in the uses of calculus to study problems involving motion, while Leibniz' work and interests involved finding extrema and solving differential equations.

4. Newton discovered the rules and processes of calculus by a study of velocity and distance, while Leibniz did so via algebraic sums and differences.

5. Newton used infinitesimals as a computational means, while Leibniz used them directly.

6. Newton's priority was differentiation while Leibniz' was integration.

7. Newton stressed the use of infinite series to express functions, while Leibniz preferred solutions that could be written in finite terms.

8. Leibniz gave more general rules and more convenient notation.

The other important writings of Newton and Leibniz reflect some of the differences mentioned above. Leibniz wrote many lengthy and influential works on philosophy, while Newton wrote several important books on the sciences. The latter include his best known work, *Philosophiæ naturalis principia mathematica (The Mathematical Principles of Natural Philosophy)*, which remains of the most important books in the sciences ever written. In this book he developed the laws of motion using calculus and used them to derive Kepler's laws of planetary motion. An extremely brief but informative summary of *Principia* is available at the following online site:

http://www.answers.com/topic/newton-s-principia

We have already mentioned some common aspects of Newton's and Leibniz' legacies with respect to calculus, and we shall conclude this discussion by mentioning some noteworthy differences:

1. Newton's applications of calculus ultimately determined the direction of subsequent work in mathematics and physics.

2. Leibniz' formulation of calculus ultimately determined how the mathematical aspects of this work were formulated (however, Newton's dot notation for derivatives is still used sometimes in physics to denote derivatives with respect to time).

Subsequent developments

We shall concentrate on points related to the material covered in first year calculus courses. Mathematics has continued to grow rapidly during the 300+ years following the invention of differential and integral calculus by Newton and Leibniz, but most of this history is well beyond the scope of the present course.

The Bernoulli family. The Bernoulli brothers – James or Jacob or Jakob or Jacques (1655 – 1705) and Jon or Johann or Jean (1667 – 1748) made numerous important contributions to the subject soon after its development and publication. One small item worth noting is that the result known as L'Hospital's Rule was originally due to John Bernoulli but was sold to G. de L'Hospital (1661 – 1704) for an influential textbook the latter published in 1696. The Bernoullis were particularly effective at applying the methods of differential and integral calculus to analyze new types of mathematical questions that had previously been out of reach. One example is the brachistochrone problem, which asks for the curve of quickest descent connecting two given points in a vertical plane; it turns out that a portion of the cycloid curve is a solution to this question. The Bernoullis made several other early contributions to the study of differential equations.

Solid analytic geometry. When plane analytic geometry was developed during the 17th century, researchers like Fermat and P. de la Hire (1640 – 1718) were convinced that one could handle questions in 3-dimensional geometry similarly by adding one more coordinate and making suitable adjustments to various formulas, but the details of this program were not completed until the 18th century. Names associated with this work include J. Hermann (1678 – 1733), A. – C. Clairaut (1713 – 1765) and L. Euler.

Infinite series. We note first that the standard infinite power series for functions are named after B. Taylor (1685 – 1731) and C. Maclaurin (1698 – 1746); the usual attributions are an accident of history through no fault of Taylor or Maclaurin.

First year calculus books always mention that the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

converges, but often the value of the sum is not mentioned. In fact, Euler proved the unexpected relationship

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

using elaborate manipulations of infinite series. This was just one in a sequence of increasingly bold and dramatic summation formulas that Euler derived. Such conclusions ultimately led Euler to carry out many speculative operations on infinite series that do **not** converge. In particular, he suggested that ½ is a reasonable value to take as the sum of the following divergent series:

$$1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) + \ldots$$

This may seem absurd, but it turns out that one can find some good mathematical justifications for attaching this value to the divergent series. However, it should not be surprising that eventually one encounters supposed formulas that lead to contradictory answers. Examples of this sort are somewhat artificial, but one can also construct important classes of physical problems involving infinite series with closely related convergence difficulties. This leads naturally to the next topic

The logical soundness of calculus

Aside from the questions on infinite series that we have just raised, there are even more important issues regarding calculus that were problematic during the 17^{th} and 18^{th} century. The most important of these was the use of infinitesimals. Not surprisingly there were many questions about the logical consistency of using objects that were smaller than any finite positive quantity but still positive. Proponents of calculus attempted to explain this concept, but such explanations didn't really make much sense to mathematicians of that time; even though the computational methods of Newton and Leibniz were getting the right answers, regardless of whether the explanations were understandable. Probably the most famous critique of infinitesimals was *The Analyst*, by Bishop G. Berkeley (1685 – 1753); mathematicians and others realized the validity of his claims. Progress in mathematics continued at a rapid pace, but Berkeley's criticisms reinforced earlier views of many that calculus needed a more secure logical foundation. With the development of calculus, mathematics had moved into new territory, not just abstracting familiar ideas but also contributing new concepts of its own. It was also rapidly accepting an ever expanding collection of ideas and methods that were increasingly removed from simple experience. In order to handle such new concepts it is necessary to maintain strict logical standards. Of course, the same applies to the elaborate manipulations with infinite series that mathematicians had been carrying out.

The resolution of the problems with infinitesimals led mathematicians to base calculus on the concept of limit. This need had already been tentatively anticipated by Wallis and Gregory. J. L. D'Alembert (1717 – 1783) proposed a definition of limits, but the wording needed to be made more precise. The decisive step in this direction was due to A. – L. Cauchy (1789 – 1857). In particular, his text of 1821 included the concept of limit (a concept which had not appeared explicitly in the work of Newton or Leibniz) in a form very close the one in use today. His definition of derivative is precisely the one used today. Cauchy also stressed that the definite integral should be defined as the limit certain algebraic sums and is independent of the definition of the derivative. It is from Cauchy's view of the integral that broad modern generalizations of this concept have developed. The modern definition of limit using δ and ε is due to K. Weierstrass (1815 – 1897).

Despite their doubtful logical status, users of mathematics continued to work with infinitesimals, probably motivated by their relative simplicity, the fact that they gave reliable answers, and an expectation that mathematicians could ultimately find a logical justification for whatever was being attempted. During the nineteen sixties Abraham Robinson (1918 – 1974) used extensive machinery from abstract mathematical logic to show that one can in fact construct a number system with infinitesimals that satisfy the usual rules of arithmetic. However, the advantage of Robinson's concept of infinitesimal – its logical soundness – is balanced by the fact that, unlike 17^{th} century infinitesimals, it is neither simple nor intuitively easy to understand.

The definition of limit was one step in strengthening the mathematical foundations for calculus. Mathematicians also came to realize that functions could behave in bizarre manners that they had not previously considered, and it was necessary to take such examples into account. Stronger logical justifications were needed for many basic points in calculus; for example, the fact that continuous functions on open intervals are bounded and the Intermediate Value Property for continuous functions. Further thought was needed to understand the problems that arose if one was too casual when working with infinite series. Such questions were not just academic, for in fact they arise quickly in connection with real world problems like studying wave motion. Ultimately a secure logical foundation for calculus required a logically rigorous description of the real number system, which in turn required a theory of infinite sets. The necessary machinery was provided by R. Dedekind (1831 – 1816) and G. Cantor (1845 – 1918).