14.A. An infinite series fallacy

Here is a particularly striking example of manipulations with infinite series that yield absurd conclusions. This example is due to Euler. Bibliographic information and further examples are discussed on pages 444–449 of Kline, *mathematical Thought from Ancient to Modern Times*.

We start with the geometric series:

$$\frac{x}{1-x} \;=\; x \;+\; x^2 \;+\; x^3 \;+\; \cdots$$

On the other hand, consider also the following series expansion:

$$\frac{x}{x-1} \;=\; \frac{1}{1-\frac{1}{x}} \;=\; 1 \;+\; \frac{1}{x} \;+\; \frac{1}{x^2} \;+\; \frac{1}{x^3} \;+\; \cdots$$

If we add these two equations we obtain the purported equation

$$0 \;=\; \frac{x}{1-x} \;+\; \frac{x}{x-1} \;=\; \cdots \;+\; \frac{1}{x^3} \;+\; \frac{1}{x^2} \;+\; \frac{1}{x} \;+\; 1 \;+\; x \;+\; x^2 \;+\; x^3 \;+\; \cdots$$

which should look suspicious for several reasons. For example, if $x = 2$ or $x = \frac{1}{2}$ then we obtain the purported identity

$$0 \;=\; \cdots \;+\; \frac{1}{2^3} \;+\; \frac{1}{2^2} \;+\; \frac{1}{2} \;+\; 1 \;+\; 2 \;+\; 2^2 \;+\; 2^3 \;+\; \cdots$$

which seems to suggest that an infinite sum of certain positive real numbers is zero!

What is wrong here? If we substitute $x = \frac{1}{2}$ into the first geometric series then we certainly get a valid result. However, the second expansion is definitely not valid if $x = \frac{1}{2}$. The reason for this failure is simple: The terms of the infinite series expansion do not go to zero as $n \to \infty$.■