14.B. An infinite series fallacy — II

Here is a more formal discussion of the fallacious infinite series formula.

Euler's argument is valid in a commutative ring with unit R that satisfies the following additional conditions:

(1) There is a unit preserving homomorphism from the finite Laurent series ring $\mathbf{Z}[x, x^{-1}]$ to R.

(2) There is a topology on R such that the geometric series $\sum_k y^k$ converge, where $y = x$ and $y = 1/x$.

The purported application of the argument requires a third condition:

(3) The algebra A of continuous functions on some closed interval J such that $\frac{1}{2} \in J \subset (0,1)$ satisfies the conditions above for some topology containing the topology of pointwise convergence.

The absurd formulas obtained in the previous document lead to a *reductio ad absurdum* proof that (3) must be false. In fact, one can show more: **CLAIM.** *There is no topological ring R satisfying* (1) *and* (2) *such that the topology on R is Hausdorff.*

Proof. The point of the Hausdorff condition is that one obtains unique limits.

Suppose that there is a topological ring as described. Since both geometric series converge we have

$$0 \quad = \quad \lim_{n \to \infty} x^n \quad = \quad \lim_{n \to \infty} \left(\frac{1}{x}\right)^n$$

and taking products we have

$$0 \quad = \quad \lim_{n \to \infty} x^n \cdot \lim_{n \to \infty} \left(\frac{1}{x}\right)^n \quad = \quad \lim_{n \to \infty} x^n \cdot \left(\frac{1}{x}\right)^n \quad = \quad \lim_{n \to \infty} 1 \quad = \quad 1\ .$$

Since $0 = 1$ this yields a contradiction, and thus there is no topological ring with the specified properties.∎