SOLUTIONS TO EXERCISES FOR

MATHEMATICS 153 — Assignment 1

Spring 2005

Problems from Burton, p. 26

3. The fraction 1/6 is equal to 10/60 and therefore the sexagesimal expression is 0;10.

To find the expansion for 1/9 we need to solve $1/9 = x/60$. By elementary algebra this means $9x = 60$ or $x = 6\frac{2}{3}$. Thus

$$x \quad = \quad \frac{6}{60} + \frac{2}{3} \cdot \frac{1}{60} \quad = \quad \frac{6}{60} + \frac{40}{60} \cdot \frac{1}{60}$$

which yields the sexagsimal expression $0;10,40$ for 1/9.

Finding the expression for 1/5 just amounts to writing this as 12/60, so the form here is 0;12.

To find 1/24 we again write $1/24 = x/60$ and solve for x to get $x = 2\frac{1}{2}$. Now $\frac{1}{2} = \frac{30}{60}$ and therefore we can proceed as in the second example to conclude that the sexagesimal form for 1/24 is 0;2,30.

One proceeds similarly for 1/40, solving $1/40 = x/60$ to get $x = 1\frac{1}{2}$. Much as in the preceding discussion this yields the form 0;1,30.

Finally, the same method leads to the equation $5/12 = x/60$, which implies that 5/12 has the sexagesimal form 0;25.■

4. We shall only rewrite these in standard base 10 fractional notation. The answers are in the back of Burton.

(a) The sexagesimal number 1,23,45 is equal to $1 \times 3600 \ + \ 23 \times 60 \ + \ 45$.

(b) This number is equal to

$$12 \ + \ \frac{3}{60} \ + \ \frac{45}{60 \times 60} \ .$$

(c) This number is equal to the previous one divided by 60.

(d) This number is simply equal to the first one divided by 60.

5. The general rule is to shift the semicolon one place to the right, so in this particular example the product is 1,23,45;6.

Additional problem 1. Express the ordinary fractions

$$\tfrac{2}{9} \ , \ \tfrac{1}{25} \ , \ \tfrac{1}{100} \ , \ \tfrac{1}{125}$$

in sexagesimal form.

SOLUTION.

We shall do these in order.

To find the sexagesimal form for 2/9 we have to write it as $x/60$. We can find x by the usual method for solving proportion equations: $9x = 2 \times 60 = 120 \Longrightarrow x = 120/9$. which translates to $x - 13\frac{1}{3}$. Now $1/3 = 20/60$, so this means we have

$$\frac{2}{9} \quad = \quad \frac{13}{60} + \frac{20}{60 \times 60} \quad = \quad 0;13,20 \ .$$

We approach $1/25$ in the same way. The solution to the equation $1/25 = x/60$ is $x = 2\frac{2}{5}$. Since $\frac{2}{5} = 24/60$ we have

$$\frac{1}{25} \quad = \quad \frac{2}{60} + \frac{24}{60 \times 60} \quad = \quad 0;2,24 \ .$$

For the next two we have fractions that are clearly less than $1/60$, so we should start with $60 \times 60 = 3600$ instead. Thus we want to start by solving $1/100 = x/3600$. This has the solution $x = 36$ and therefore the sexagesimal form is $0;0,36$.

Finally, for the last one we begin by solving $1/125 = x/3600$, and we find in this case that $x = 28\frac{4}{5}$. Now $4/5 = 48/60$ so we must have

$$\frac{1}{125} \quad = \quad \frac{28}{60 \times 60} + \frac{48}{60 \times 60 \times 60} \quad = \quad 0;0;28,48 \ .$$

Additional problem 2. The number 1;24,51,10 appears on a Babylonian cuneiform tablet. Express it on more familiar terms.

SOLUTION.

By definition this is equal to

$$1 + \frac{24}{60} + \frac{51}{60 \times 60} + \frac{10}{60 \times 60 \times 60} \quad = \quad 1 + \frac{24 \times 3600 \ + \ 51 \times 60 \ + \ 10}{60 \times 60 \times 60} \quad =$$

$$1 + \frac{86400 \ + \ 3060 \ + 10}{60 \times 60 \times 60} \quad = \quad 1 + \frac{89470}{216000} \ .$$

If we compute this number in decimals, we see that it is equal to 1.41421296296296296... and if we compare this to $\sqrt{2} = 1.4142356...$ we see that this must have been an approximation to the square root of 2 that is accurate to four (of our) decimal places.■

Additional problem 3. Prove the assertion in the notes that a rational number r satisfying $0 < r < 1$ has only finitely many Egyptian fraction expansions with a fixed length $L > 1$. [*Hint:* Proceed by induction on the length. What happens if $L = 1$? Suppose that the result is known for length L and proceed to length $L + 1$. If one has an Egyptian fraction expansion of length $L + 1$, why must one of the summands be greater than $r/(L+1)$? Show this implies that the denominator of at least one summand is $\leq (L + 1)/r$. For each positive integer $m < (L + 1)/r$ why do the induction hypothesis and the condition $0 < \frac{1}{m} < r$ imply that the fraction $r - \frac{1}{m}$ has only finitely many Egyptian fraction expansions of length L? How can one conclude the proof using this information?]

SOLUTION.

Let's start with $L = 2$. How many ways are there of writing a fraction r satisfying $0 < r < 1$ as a sum of two unit fractions? If we have an equation of the form

$$r \quad = \quad \frac{1}{a} + \frac{1}{b}$$

where for the sake of definitess we shall take $a < b$, then we have inequalities

$$r \quad > \quad \frac{1}{a} \quad > \quad \frac{r}{2}$$

which imply that

$$\frac{1}{r} \;<\; a \;<\; \frac{2}{r}\,.$$

Now there are only finitely many choices of integers a for which this inequality is true. For each such a consider the remainder $r - \frac{1}{a}$. This also lies between 0 and 1. If it is a unit fraction of the form $1/b$, then we have an Egyptian fraction expansion

$$r \;=\; \frac{1}{a} + \frac{1}{b}$$

but then again the remainder need not have this form. Regardless of whether or not it does, for each choice of a there is at most one way of writing r in Egyptian form such that $1/a$ is one of the terms. This means that there can only be finitely many Egyptian fraction expansions whose length L is equal to 2.

To prove the result for all L using finite induction, we need to show that if the conclusion is true for expansions of length L then it is true for expansions of length $L+1$. So suppose we have a value of L for which the conclusion is known to be true. If we are given an Egyptian expansion of r with length $L+1$, one of the terms in this expansion, say $1/a$, is larger than all the others. As in the case $L=2$, this means that

$$\frac{1}{a} \;>\; \frac{r}{L+1}$$

for otherwise every one of the summands would be less than or equal to the right hand side, and only one of the $L+1$ terms could be equal to this. In such a situation the entire sum must be strictly less than r. Armed with the above inequality for the largest term in the expansion, we proceed as follows: Combining the displayed inequality with the basic relation $\frac{1}{a} < r$, we conclude that

$$\frac{1}{r} \;<\; a \;<\; \frac{L+1}{r}$$

and see that there are only finitely many possibilities for the largest term in an Egyptian expansion of length $L+1$. As before, for each such a consider the remainder $r - \frac{1}{a}$. This also lies between 0 and 1. By the hypothesis on expansions of length L, for each choice of a there are only finitely many ways of expanding the remainder as an Egyptian fraction of length L.

Suppose now that we fix a and consider the finite collection of expansions

$$r \;=\; \frac{1}{a} + \sum_{j=1}^{L} \frac{1}{n_j}$$

given by the previous paragraph. Every Egyptian fraction expansion of r containing the term $1/a$ is in this list; in fact, there may be other non-Egyptian type expansions in the list because it is possible that there is a summand $\frac{1}{a}$ in the sum of L terms, but in any case we see that there are only finitely many ways of writing r as an Egyptian fraction of length $L+1$ such that $1/a$ is one of the terms. But there are only finitely many options for a, so this means there can only be finitely many ways of expressing r by an Egyptian fraction expansion of length $L+1$. This proves the inductive step, and therefore the conclusion is true for all $L \geq 2$.■

Problems from Burton, p. 48

8. This is simply a matter of verifying an identity and checking it against the list in the book, The expression for $\frac{2}{7}$ follows from $7+1=8$, the expression for $\frac{2}{35}$ follows from $35=7\times 5$ and $7+5=12$, and finally the expression for $\frac{2}{91}$ follows from $91=13\times 7$ and $7+13=20$.■

13. This uses the identity

$$\frac{1}{m} \quad = \quad \frac{1}{m+1} \quad + \quad \frac{1}{m(m+1)}$$

which can be checked directly. If n divides $m+1$ this means that $m+1=n\cdot q$ for some q, If we multiply both sides of the displayed equation by n, substitute the factorization of $m+1$ into the equation and simplify, we then obtain the identity

$$\frac{n}{m} \quad = \quad \frac{1}{q} \; + \; \frac{1}{q\,m} \; .■$$

Additional problem 4. Express $\frac{p}{11}$ as an Egyptian fraction for each p such that $2\leq p\leq 10$.

SOLUTION.

We shall try to do the first few these using the Greedy Algorithm.

The largest unit fraction less than $\frac{2}{11}$ can be found by looking for the first integer which is greater than the reciprocal $\frac{11}{2}=5\frac{1}{2}$. This integer is 6. Therefore the Greedy Algorithm gives $\frac{1}{6}$ as the first term and proceeds to consider the remainder. But $\frac{2}{11}-\frac{1}{6}=\frac{1}{66}$ so the we obtain the expansion $\frac{2}{11}=\frac{1}{6}+\frac{1}{66}$ right away.

Next consider $\frac{3}{11}$. In this case the Greedy Algorithm gives $\frac{1}{4}$ as the first term, and we compute the remainder to be $\frac{3}{11}-\frac{1}{4}=\frac{1}{44}$, so that $\frac{3}{11}=\frac{1}{4}+\frac{1}{44}$ in this case.

Now consider $\frac{4}{11}$, in which case the Greedy Algorithm yields $\frac{1}{3}$ as the first term and the remainder is $\frac{1}{33}$. Thus we have $\frac{4}{11}=\frac{1}{3}+\frac{1}{33}$ in this case.

In the case of $\frac{5}{11}$, the Greedy Algorithm still yields $\frac{1}{3}$ as the first term and the remainder is $\frac{4}{33}$. The latter is equal to $\frac{1}{11}+\frac{1}{33}$, and thus we have $\frac{5}{11}=\frac{1}{3}+\frac{1}{11}+\frac{1}{33}$ in this case.

For $\frac{3}{11}$, the Greedy Algorithm yields $\frac{1}{2}$ as the first term and the remainder is $\frac{1}{22}$. Thus we have $\frac{6}{11}=\frac{1}{2}+\frac{1}{22}$ in this case.

Turning to $\frac{7}{11}$ a first application of the Greedy Algorithm yields $\frac{7}{11}=\frac{1}{2}+\frac{3}{22}$. Rather than proceed to apply the Greedy Algorithm directly to the remainder of $\frac{3}{22}$, let's take the expansion we had for $\frac{3}{11}$ and multiply it by $\frac{1}{2}$ to obtain $\frac{3}{22}=\frac{1}{8}+\frac{1}{88}$. We then get the expansion $\frac{7}{11}=\frac{1}{2}+\frac{1}{8}+\frac{1}{88}$ in this case.

We can dispose of the remaining cases similarly. For $\frac{8}{11}$, combine $\frac{8}{11}=\frac{1}{2}+\frac{5}{22}$ and $\frac{5}{22}=\frac{1}{6}+\frac{1}{22}+\frac{1}{66}$ to obtain $\frac{8}{11}=\frac{1}{2}+\frac{1}{6}+\frac{1}{22}+\frac{1}{66}$ in this case.

Similarly, for $\frac{9}{11}$, combine $\frac{9}{11}=\frac{1}{2}+\frac{7}{22}$ and $\frac{7}{22}=\frac{1}{4}+\frac{1}{16}+\frac{1}{176}$ to obtain $\frac{9}{11}=\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{176}$.

Finally, for $\frac{10}{11}$, combine $\frac{10}{11}=\frac{1}{2}+\frac{9}{22}$ and $\frac{9}{22}=\frac{1}{4}+\frac{1}{8}+\frac{1}{32}+\frac{1}{352}$ to obtain $\frac{10}{11}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{32}+\frac{1}{352}$.■

Problem from Burton, p. 59

9. (a) If we apply the formula in the exercise to each of the triangles $\triangle ADC$, $\triangle DCB$, $\triangle CBA$, $\triangle BAD$, we find that the sum of their areas is

$$\tfrac{1}{2}\,dc\,\sin\,D \;+\; \tfrac{1}{2}\,cb\,\sin\,C \;+\; \tfrac{1}{2}\,ba\,\sin\,B \;+\; \tfrac{1}{2}\,ad\,\sin\,A\;.$$

Let X be the point where AC and BD meet. Then we have the following equations:

$$\begin{aligned}
\textbf{area}(\triangle ADC) &= \textbf{area}(\triangle AXD) + \textbf{area}(\triangle DXC) \\
\textbf{area}(\triangle DCB) &= \textbf{area}(\triangle DXC) + \textbf{area}(\triangle CXB) \\
\textbf{area}(\triangle CBA) &= \textbf{area}(\triangle CXB) + \textbf{area}(\triangle BXA) \\
\textbf{area}(\triangle BAD) &= \textbf{area}(\triangle BXA) + \textbf{area}(\triangle AXD)
\end{aligned}$$

Now we also know that the area $\mathbf{A}$ of the quadrilateral $ABCD$ is equal to

$$\textbf{area}(\triangle DXC) + \textbf{area}(\triangle CXB) + \textbf{area}(\triangle BXA) + \textbf{area}(\triangle AXD)$$

and if we add the previous four lines we find that the sum of the areas of $\triangle ADC$, $\triangle DCB$, $\triangle CBA$ and $\triangle BAD$ is equal to $2\,\mathbf{A}$. Substituting this into the first expression in this exercise we obtain the formula

$$2\,\mathbf{A} = \tfrac{1}{2}\,dc\,\sin D + \tfrac{1}{2}\,cb\,\sin C + \tfrac{1}{2}\,ba\,\sin B + \tfrac{1}{2}\,ad\,\sin A$$

and if we divide both sides of this equation by 2 we obtain the area formula given in the exercise.■

(b) Since the sine of an angle between $0°$ and $180°$ is ≤ 1, the formula in (a) implies that

$$\mathbf{A} \leq \tfrac{1}{4}(dc + cb + ba + ad)$$

and the inequality in the exercise follows because the right hand side is equal to $\frac{1}{4}\,(a+c)\,(b+d)$. Furthermore if any of the vertex angles $\angle A$, $\angle B$, $\angle C$, $\angle D$ is NOT a right angle, then the sine of that angle is strictly less than one and this implies the inequality must be strict.■

Problems from Burton, p. 67

4. We need to solve the equations $x + y = 10$ and $xy = 16$ for x and y. Given that this is a second degree system we can expect to find two solutions but for a meaningful solution of the original physical problem both x and y must be positive.

The hint suggests using the formula

$$(x-y)^2 = (x+y)^2 - 4\,xy$$

and if we substitute the given equations into the right hand side we obtain the equation

$$(x-y)^2 = 10^2 - 4\,16 = 36\ .$$

Thus we have $x - y = \pm\,6$. If $x - y = +6$, the solution we obtain is $x = 8, y = 2$, while if $x - y = -6$ we obtain the solution $y = 8, x = 2$. In particular, this means that we have a rectangle that is 8 by 2.

6. (a) In this case use the formula

$$(x-y)^2 = (x+y)^2 - 4\,xy$$

to find $(x+y)^2$. Specifically, we have $36 = (x+y)^2 - 64$, which leads to $x + y = \pm\,10$, whose solutions are $(x, y) = (6, 2)$ and $(2, 6)$.■

(b) Use the same formula as before, but substitute the numerical values for this specific problem to obtain the equation $16 = (x+y)^2 - 84$. Once again this leads to $x+y = \pm 10$, and the solutions now are $(x,y) = (7,3)$ and $(3,7)$.■

(c) Once again use the formula, this time obtaining the equation $(x-y)^2 = 64 - 60 = 4$. Thus we have $x+y = 2$, and the solutions for this problem are $(x,y) = (5,3)$ and $(3,5)$.■

Burton, p, 67: 4, 6, 13abc

13. (a) The hint on page 68 of the text seems wrong, and since one already knows x it is reasonable to approach this by substituting the first equation into the second. This yields the equation

$$30\,y \;-\; (30-y)^2 \;=\; 500$$

which after expansion and simplification reduces to

$$y^2 \;-\; 90\,y \;+\; 1400 \;=\; 0\,.$$

The roots of this equation are $y = 70$ and $y = 20$, and as noted before we are given $x = 30$.■

(b) Here we follow the hint on page 68 of the text and subtract the square of the first equation from twice the second. The square of the first equation has the form $x^2 + 2\,xy + y^2 = 2500$. If we subtract this from $2\,x^2 + 2\,y^2 + 2\,(x-y)^2 = 2800$ we obtain the following:

$$3\,(x-y)^2 \;=\; x^2 \;-\; 2\,xy \;+\; y^2 \;+\; (x-y)^2 \;=\; 300$$

This implies $x - y = \pm 10$. Combining these with the original equation $x + y = 50$, we obtain the solutions $(x,y) = (30,20)$ and $(20,30)$ depending upon the sign $\pm$.■

(c) In this problem we also follow the hint and substitute $(x+y)^2 = (x-y)^2 + 4\,xy$ and $xy = 600$ into the equation $(x+y)^2 + 60\,(x-y) = 3100$. This yields the following quadratic equation in $(x-y)$:

$$(x-y)^2 \;+\; 60\,(x-y) \;-\; 700 \;=\; 0$$

The roots of this equation are $x - y = 10$ and -70. If we substitute this into $xy = 600$ and solve we obtain the solutions $(x,y) = (30,20)$ and $(-20,-30)$ when $x - y = 10$, and when $x - y = -70$ we obtain the solutions $(x,y) = (-35 - 5\sqrt{73},\, 35 - 5\sqrt{73})$ and $(-35 + 5\sqrt{73},\, 35 + 5\sqrt{73})$.■

Additional problem 5. Exercise 5(b) on page 75 of Burton mentions an incorrect Babylonian formula for the area of an isosceles trapezoid $ABCD$:

$$\text{area} \;=\; \frac{(a+c)\cdot(b+d)}{4}$$

Here a and c are the lengths of the two parallel sides and b and d are the lengths of the nonparallel sides. The files `trapezoidABCD.*` — where $*$ = `ps`, `pdf` or `jpg` — give an illustration that is consistent with Exercise 9 on page 58 of Burton.

(i) Using the formula in the exercise on page 58, find the actual area of an isosceles trapezoid in term of the lengths of the sides and trigonometric functions of the angle θ at the vertex A.

[Recall that the nonparallel sides have equal length, the measures of the vertex angles at A and B are equal, and the measures of the vertex angles at C and D are supplementary to those at A and B; recall that two angles are supplementary if their measures add up to $180°$.]

SOLUTION.

Since supplementary angles have the same sines and all vertex angles are either supplementary to $\angle\, A$ or have the same measure as $\angle\, A$, it follows that $\sin\, A = \sin\, B = \sin\, C = \sin\, D$. Thus the formula reduces to

$$\mathbf{A} \;=\; \tfrac{1}{4}\,(a+c)\,(b+d)\,\sin\,\theta\,.$$

(ii) What is the ratio of the actual area to the figure given by the formula if the vertex angles at A and B are $60°$ angles?

SOLUTION.

The formula says that the ratio of the actual area to the formula area is equal to $\sin\theta$. For a $60°$ angle this sine is equal to $\frac{1}{2}\sqrt{3}$, and therefore the ratio of the actual to formula area in this case is also equal to $\frac{1}{2}\sqrt{3}$. This is approximately 87 per cent of the value predicted by the incorrect formula.■