# SOLUTIONS TO EXERCISES FOR

# MATHEMATICS 153 — Assignment 2

**Spring 2005**

## PROBLEMS FROM BURTON, p. 98

**3.** By definition $t_n = (n^2+n)/2$. We need to show that $9\,t_n + 1$ is equal to $t_m$ for some $m$. Let's see what happens if we substitute the expression for $t_n$ into $9\,t_n + 1$.

$$9\left(\frac{n^2+n}{2}\right) + 1 \quad = \quad \frac{9n^2+9n+2}{2} \quad = \quad \frac{(3n+1)(3n+2)}{2} \quad = \quad t_{3n+1}$$

**6.** The number 1225 is equal to $35^2$, and we follow the hint to write it as a triangular number $t_n$. According to the hint $n$ is a root of the quadratic equation $n^2+n-2450 = 0$. The quadratic formula implies that the roots of the latter are equal to

$$\frac{-1 \pm \sqrt{1+4\cdot 2450}}{2} \quad = \quad \frac{-1 \pm 99}{2}$$

and hence 49 is the unique positive root. Therefore we see that $1225 = t_{49}$.■

Similarly, we check that $41616 = 204^2$ by some method, maybe using a calculator or table, maybe educated guessing, maybe the old algorithm for finding square roots that used to be taught in elemetary schools (which might be discussed later in this course). We must then find the positive root of the quadratic equation $n^2+n-83232 = 0$, and this can be done using the quadratic formula:

$$n \quad = \quad \frac{-1 \pm \sqrt{1+4\cdot 83232}}{2} \quad = \quad \frac{-1 \pm \sqrt{332929}}{2} \quad = \quad \frac{-1 \pm 577}{2}$$

The positive root of this pair is 288, and therefore we have shown that $41616 = t_{288}$.■

**11.** $(c)$ Follow the hint and write things out as

$$\sum_{k=1}^{n} k(k+1) \quad = \quad \sum_{k=1}^{n} k^2 + k \quad = \quad \sum_{k=1}^{n} k^2 \; + \; \sum_{k=1}^{n} k \quad =$$

$$\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \quad = \quad \frac{n(n+1)(n+2)}{3}\,.\blacksquare$$

$(d)$ Once again we use the hint:

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} \quad = \quad \frac{1}{2}\cdot\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) \quad =$$

$$\frac{1}{2}\cdot\left(1 - \frac{1}{2n+1}\right) \quad = \quad \frac{n}{2n+1}$$

This is an example of a telescoping series. The second piece of the $k^{\text{th}}$ term cancels the first piece of the next term for every $k \leq n-1$.■

**PROBLEMS FROM BURTON, p. 112**

**2.** Follow the hint, noting that if$x^2+y^2=z^2$ and $x+y+z=\frac{1}{2}\,xy$, then $(x-4)(y-4)=8$. One could try to set up a system of quadratic equations, but in a simple case like this it is better to do things by hand. Since we are looking for integral solutions, this means that each of $x-4$ and $y-4$ must be 1, 2, 4 or 8. Thus the possible solutions to check are $(x,y)=(5,12),(6,8),(8,6)and(12,5)$. In the first and last cases we have $z=13$, while in the middle two we have $z=10$. Which of these choices for $(x,y,z)$ also satisfy the equation $x+y+z=\frac{1}{2}\,xy$? Direct substitution shows that they all do.■

**3.** Following the hint, if $n$ is odd we need to check that

$$n^2 \;+\; \left(\frac{1}{2}(n^2-1)\right)^2 \;=\; \left(\frac{1}{2}(n^2+1)\right)^2$$

and if $n$ is even we need to check that

$$n^2 \;+\; \left(\frac{n^2}{4}-1\right)^2 \;=\; \left(\frac{n^2}{4}+1\right)^2 .$$

We do these as follows:

$$n^2 \;+\; \frac{n^4-2n^2+1}{4} \;=\; \frac{n^4+2n^2+1}{4} \;=\; \left(\frac{1}{2}(n^2+1)\right)^2$$

$$n^2 \;+\; \left(\frac{n^2}{4}-1\right)^2 \;=\; n^2 \;+\; \frac{n^4}{16} \;-\; \frac{n^2}{2} \;+1 \;=$$

$$\frac{n^4}{16} \;+\; \frac{n^2}{2} \;+1 \;=\; \left(\frac{n^2}{4}+1\right)^2 .$$

**7.** If we divide the equation $y_n^2-2\,x_n^2=1$ by $x_n 2$ we obtain

$$\left(\frac{y_n}{x_n}\right)^2 \;-\; 2 \;=\; \frac{1}{x_n^2} .$$

This means that $y_n\,/\,x_n$ will approach $\sqrt{2}$ in the limit if we know that $x_n \longrightarrow \infty$. However, by construction we know tht $x_n, y_n \geq 0$, and therefore the recursive relation $x_n = 3\,x_{n-1}+2\,y_{n-1}$ implies $x_n \geq 3^{n-1}x_1 = 2\cdot 3^{n-1}$. Thus $x_n$ grows exponentially with $n$ and as noted before this implies that the sequence of fractions $y_n\,/\,x_n$ has a limit which is equal to $\sqrt{2}$.■

**18.** We first show that $h$ satisfies the harmonic mean equation if and only if it satisfies $(b)$:

$$\frac{h-a}{b-h} \;=\; \frac{a}{b} \iff b(h-a) \;=\; a(b-h) \iff h(a+b) \;=\; 2\,ab \iff$$

$$h \;=\; \frac{2\,ab}{a+b}$$

Next we show that $h$ satisfies $(a)$ if and only if it satisfies the harmonic mean equation:

$$\frac{1}{a} - \frac{1}{h} = \frac{1}{h} - \frac{1}{b} \iff \frac{h-a}{ha} = \frac{b-h}{hb} \iff$$

$$\frac{h-a}{b-h} = \frac{ha}{hb} = \frac{a}{b}$$

**PROBLEMS FROM BURTON, p. 123**

**4.** Since the point $(x, y)$ lies on the intersection of the two parabolas we know that $y^2 = 2\,ax$ and $x^2 = ay$. If we square the second equation we obtain $x^4 = a^2y^2 = a^2\,(2\,ax) = 2\,a^3x$. The real solutions of this equation are $x = 0$ and $x = \mathbf{cbrt}(2) \cdot a$. Since the intersection point has a positive first coordinate, the second solution is the one we want.■

**PROBLEMS FROM BURTON, p. 478**

**3.** $(a)$ The sum of all the proper divisors of $p^n$ is

$$\sum_{k=0}^{n-1} p^k = \frac{p^n - 1}{p-1}$$

so if $p^n$ were a perfect number then the right hand side would be equal to $p^n$. Multiplying both sides of such an equation by $(p-1)$ clears out the fractions and yields $p^n - 1 = (p-1) \cdot p^n$. Now the right hand side is at least $p^n$, so this inequality cannot hold and we have a contradiction. The problem arises because we assumed $p^n$ was perfect, and therefore this cannot be true.■

$(b)$ Following the hint, if $p$ and $q$ are odd primes then $p-1$ and $q-1$ are at least 2 so that

$$(p-1) \cdot (q-1) \geq 2 \cdot 2 = 4 .$$

If we expand the left hand side we obtain

$$pq - p - q + 1 \geq 4$$

which shows that $pq > p + q + 1$. Therefore $p\,q$ cannot be a perfect number.■

**6.** Since $n$ is perfect we know that

$$\sum_{d|n} d = 2n$$

and if we divide everything by $n$ we obtain

$$\sum_{d|n} \frac{d}{n} = 2 .$$

We now make a change of variables, setting $e = n/d$. As $d$ runs through all divisors of $n$, the new variable $e$ also runds through all divisors of $n$, so we may rewrite the second sum as

$$\sum_{e|n} \frac{1}{e} = 2 .$$

**11.** To see that $p$ cannot be part of an amicable pair, notice that it only has one proper divisor; namely, 1. Thus the sum of the proper divisors is 1, and since the sum of the proper divisors of 1 is 0 this means that $p$ and 1 do not form an amicable pair.

Turning to $p^2$, since we know its proper divisors are 1 and $p$, the other number in the amicable pair would have to be $p+1$. Thus we need to show that the sum of the proper divisors of $p+1$ is not equal to $p^2$. Now the largest proper divisor is $\frac{1}{2}(p+1)$. This has two consequences. First, this is an upper bound on the size of any proper divisor. Second, there are at most $\frac{1}{2}(p+1)$ proper divisors of $p+1$. Thus the sum of the proper divisors is less than or equal to the upper bound on the number of proper divisors times the upper bound on the size of all such divisors. Upper bounds for both are given by $\frac{1}{2}(p+1)$, so a bound for the sum of the proper divisors is given by this number times itself, which is $\frac{1}{4}(p+1)^2$, If $p^2$ and $p+1$ were an amicable pair, this would mean that $p^2$ would be the sum of all the proper divisors of $p+1$ and consequently would be less than or equal to $\frac{1}{4}(p+1)^2$. However, if $p$ is an odd prime then $p > \frac{1}{2}(p+1)$, and squaring this equation yields

$$p^2 > \frac{1}{4}(p+1)^2$$

which contradicts our previous conclusion. This means that $p^2$ cannot be part of an amicable pair.∎