SOLUTIONS TO EXERCISES FOR
MATHEMATICS 153 — Assignment 7

Spring 2005

PROBLEMS FROM BURTON, p. 264

7. The condition implies $2n-1=u^2$, and we have

$$n^2 \;=\; \big((n-1)+1\big)^2 \;=\; (n-1)^2+2(n-1)+1 \;=\; (n-1)^2+(2n-1) \;=\; (n-1)^2+u^2\,.$$

Examples can be given using odd numbers that are perfect squares.■

11. Let x and y be the lengths of the legs of the right triangle, so that $b^2=\frac{1}{2}xy$ and $a^2=x^2+y^2$. Then we have $a^2+4b^2=(x+y)^2$ and $a^2-4b^2=(x-y)^2$. If we solve these for x and y we obtain the values

$$\tfrac{1}{2}\left(\sqrt{a^2+4b^2}\;\pm\;\sqrt{a^2-4b^2}\right)$$

as asserted in the problem.■

PROBLEM FROM BURTON, p. 272

2. (a) Follow the hint to add the inequalities $F_{2i+1}=F_{2i}-F_{2i-2}$ for $1\le i\le n$. The left hand side will be the sum of the odd Fibonacci numbers from F_1 to F_{2n-1}, and the right hand side is a telescoping sum whose value will be $F_{2n}-F_0=F_{2n}$.■

(c) The formula is true for $n=1$ so proceed by induction and assume it is true for $k\le n-1$. Then we have

$$\sum_{i=1}^{n}(-1)^{i+1}F_i \;=\; \sum_{i=1}^{n-1}(-1)^{i+1}F_i \;+\; (-1)^{n+1}F_n \;=\; (-1)^nF_{n-2}+1+(-1)^{n+1}F_n$$

and since $F_{n-1}=F_n-F_{n-2}$ the right hand side reduces to

$$1+(-1)^{n+1}F_{n-1}$$

completing the inductive step of the proof.■

PROBLEMS FROM BURTON, p. 302

1. (b) Set $y=x+2$, so that

$$2 \;=\; x^3+6x^2+3x \;=\; (x+2)^3-12x-8+3x \;=\; (x+2)^3-9(x+2)+18-8=y^3-9y+10$$

and hence the original polynomial reduces to $y^3-9y+8=0$. One root of this polynomial is $y=1$, and $y^3-9y+8=(y-1)(y^2+y-8)$, so the remaining roots are $-\frac{1}{2}(1\pm\sqrt{33})$. Using the inverse substitution $x=y-2$ we see that the roots of the original polynomial are -1 and $-\frac{1}{2}(5\pm\sqrt{33})$.■

3. (d) We shall try to use Cardan's formula as printed on page 302. In our situation $p=9$ and $q=12$. Thus

$$\frac{q}{2} \;=\; 6\quad,\quad \frac{q^2}{4}\;=\;36\quad,\quad \frac{p^3}{27}\;=\;27$$

, so the formula yields the root value

$$\sqrt[3]{6+\sqrt{36-27}} \;+\; \sqrt[3]{6-\sqrt{36-27}} \;=\; \sqrt[3]{6+\sqrt{9}} \;+\; \sqrt[3]{6-\sqrt{9}} \;=$$

$$\sqrt[3]{6+3} \;+\; \sqrt[3]{6-3} \;=\; \sqrt[3]{9} \;+\; \sqrt[3]{3}\,.$$