Solutions to review problems for midterm

1. The proper divisors of $10p$ are

$$1, 2, 5, 10 \quad p, 2p, 5p$$

and their sum is

$8p + 18$, which is less than $10p$ if $p \geq 11$ ($2p \geq 22 > 18$)

Also, the proper divisors of pq are $1, p, q$, so we need to show $pq \neq p+q+1$. By the hint we consider $(p-1)(q-1)$. Since one of p, q is at least 3 and the other is at least 5, the product is at least 8. Therefore

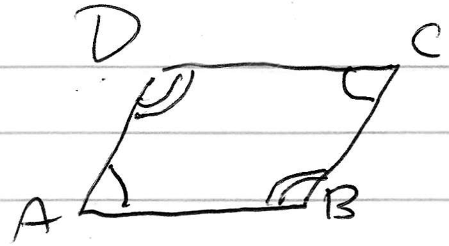
$(p-1)(q-1) = pq - p - q + 1 \geq 8$, so that

$pq \geq p + q + 7 > p + q + 1$.

2. The point is to ~~find~~ show that there are no odd primes p, q such that

$$1 + p = q^2, \quad 1 + q = p^2$$

If so, then $p^2 - q^2 = q - p$. This is impossible if $p \neq q$. Say $q > p$. Then the lefthand side is negative but the right is positive. Similarly if $p > q$.

3. 

parallelogram

If it is inscribed in a circle then $|\measuredangle A| + |\measuredangle C| = 180° = |\measuredangle B| + |\measuredangle D|$. But in a parallelogram we have $|\measuredangle A| = |\measuredangle C|$ and $|\measuredangle B| = |\measuredangle D|$, so the first two equations reduce to

$$2|\measuredangle A| = 2|\measuredangle C| = 180° = 2|\measuredangle B| = 2|\measuredangle D|$$

so that $|\measuredangle A| = |\measuredangle C| = 90° = |\measuredangle B| = |\measuredangle D|$. and hence a parallelogram inscribed in a circle must be a rectangle.

Note that, conversely, every rectangle can be inscribed in a circle whose center is the point where the diagonals meet.

4. $\mathcal{C}$ is the circle, and $(\frac{1}{2}, 0)$ lies on $y = mx + b$ where $m = 0$. Hence $0 = m \cdot \frac{1}{2} + b$, so $\cancel{b = -\frac{1}{2}m}$ $b = -\frac{1}{2}m$

So we want to find the points with $y = m(x - \frac{1}{2})$ such that $x^2 + y^2 = 1$.

The solutions to these equations are

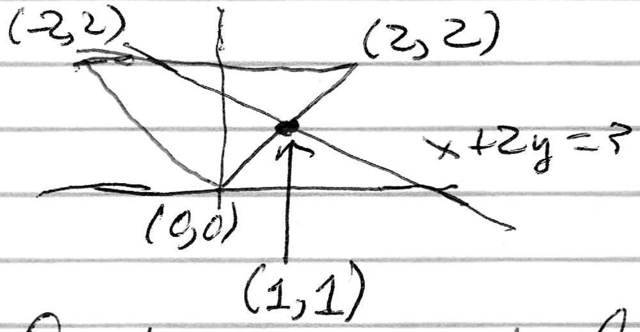
$$x = \frac{m^2 \pm \sqrt{m^4 + 5}}{2(1 + m^2)} \qquad y = ??$$

One can check that one value of x is greater than $\frac{1}{2}$ while the other is less; one can check this directly if, say, $m=2$.

~~Now $y>0$ if $x<\frac{1}{2}$ and $y<0$ if~~

Now $y<0$ if $x<\frac{1}{2}$ and $y>0$ if $x>\frac{1}{2}$ because $y=m(x-\frac{1}{2})$, so it follows that the two points which solve the system of equations are such that $y_1<0<y_2$, which means that $P=(\frac{1}{2},0)$ lies between them.

5.



The drawing suggests the conclusion. Solve the system

$$x+2y=3$$
$$y=2$$

We get $(x,y)=(-1,2)$, and this point lies on $y=2$, and it is between $(-2,2)$ and $(2,2)$ because $-2<-1<2$.