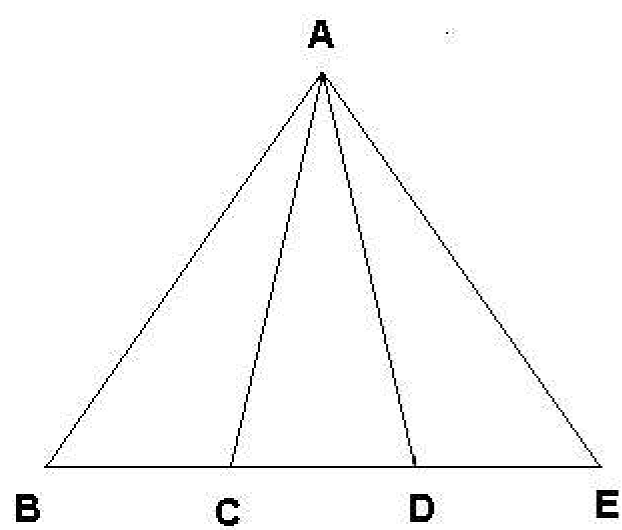
2.C. An Easy Trisection Fallacy

We have already noted that any purported straightedge and compass construction for trisecting an angle will be incorrect. The following simple example illustrates how appealing such a construction might appear at first and how one can look more closely to find a mistake.

Suppose we are given an angle ∠ **BAE** as in the diagram below and we wish to trisect it. Let's assume that the lengths of the segments **BA** and **AE** are equal. It is known that segments can be divided into any number of pieces of equal length by straightedge and compass, so apply this to segment **BE** and divide it into three equal segments that we shall call **BC**, **CD** and **DE**. If we look at the picture it might seem that **AC** and **AD** trisect ∠ **BAE**, but is this really true?



One can use the classical methods of Euclidean geometry to conclude that the segments **AC** and **AD** have equal length, and it is possible to analyze this figure even further using classical methods, but we shall take a shortcut using trigonometry.

Let **h** denote the common altitude of the isosceles triangles **BAE** and **CAD**, and let **|XY|** denote the length of the segment joining **X** and **Y**. Then standard results in trigonometry imply the following relationships:

$$\tan \tfrac{1}{2} \angle BAE = |BE| / 2h \qquad \tan \tfrac{1}{2} \angle CAD = |CD| / 2h = |BE| / 6h$$

From these formulas we conclude that **tan ½ ∠ CAD** is one third of **tan ½ ∠ BAE.** If this construction yielded a trisection then we would have a trigonometric identity of the form

$$(\tan x) / 3 = \tan (x / 3)$$

and one can check directly from tables (or a scientific calculator) that the first expression is always greater than the second (and the middle angle is always larger than the others).