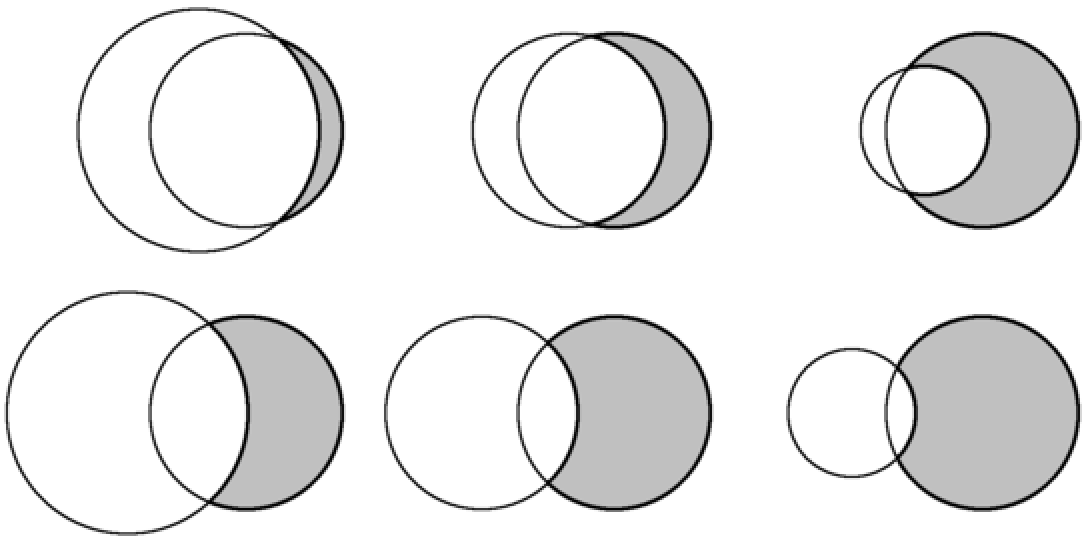
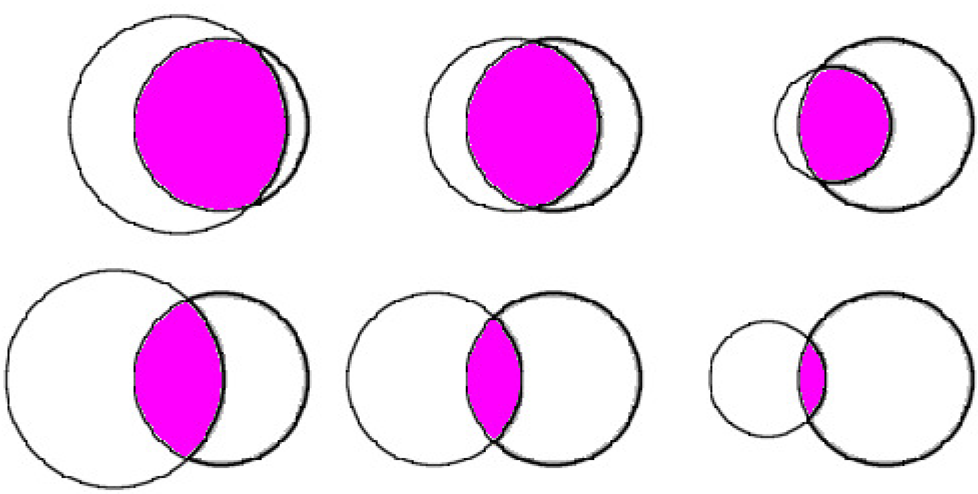
2.D. Areas of Lunes

The result of Hippocrates of Chios on areas of lunes was actually one of several such formulas which Hippocrates discovered. For the sake of completeness, we note that a ***lune*** in the plane is the region bounded by a pair of circular arcs which meet at their two endpoints. In the illustration below, each of the regions shaded in gray is a lune. If both arcs lie on the same side of the line joining their endpoints, then the lune is ***crescent shaped*** and in many cases looks like a phase of the moon. The shaded regions below are all crescent shaped lunes.



(**Source:** Lune (mathematics) cited in http://en.wikipedia.org/wiki/Lune)

On the other hand, if the two arcs lie on opposite sides of the line joining their endpoints, then the lune is said to be ***gibbous*** (JIB – us, from the Latin word for *hump*). In the following variation on the preceding drawing, the gibbous lunes are shaded in pink:



The results of Hippocrates showed that three types of lunes had areas that could be evaluated by relatively simple expressions; more precisely, the areas were given by numbers that were constructible using straightedge and compass (see http://math.ucr.edu/~res/history02b.pdf). Much later work in the 18th century by M. J. Wallenius (1731 – 1773) described two additional types of lunes whose areas are also ***constructible numbers*** of this type, and subsequent theorems of N. G. Chebotarev (1894 – 1947) and A. V. Dorodnov from the second quarter of the 20th century showed that there were no other types of lunes with areas that could be described in this manner. Two online references with further information are given below. The first describes the five types of lunes whose areas are given by constructible numbers, while the second discusses the work of Chebotarev and is at a more advanced level:

http://www.mathpages.com/home/kmath171.htm

http://websites.math.leidenuniv.nl/algebra/chebotarev.pdf

There also is a detailed discussion of this topic at a fairly elementary level in Chapter 10 of the following book:

T. Dantzig, *Mathematics in Ancient Greece* (Reprint of the 1955 book, *The Bequest of the Greeks*). Dover, New York, 2006.

The original edition of this book is also available at the following online site:

http://ia310936.us.archive.org/1/items/bequestofthegree032880mbp/bequestofthegree032880mbp.pdf