4. Alexandrian mathematics after Euclid — III

Due to the length of this unit, it has been split into three parts. This is the final unit, and it deals with other Greek mathematicians and scientists from the period.

The work or Archimedes and Apollonius represents the deepest work in Greek geometry, and indeed they pushed the classical methods to their limits. More powerful tools would be needed to make further advances, and these were not developed until the seventeenth century. Subsequent activity in ancient Greek mathematics was more directed towards developing the trigonometry and spherical geometry needed to do observational astronomy and studying questions of an arithmetic nature.

Eratosthenes of Cyrene

Eratosthenes (276 – 197 B.C.E.) probably comes as close as anyone from this period to reaching the levels attained by Euclid, Archimedes and Apollonius. He is probably best known for applying geometric and trigonometric ideas to estimate the diameter of the earth; this work is summarized on pages179 – 181 of Burton. Within mathematics itself, his main achievement was to give a systematic method for finding all primes which is known as the *sieve of Eratosthenes*. The idea is simple – one writes down all the numbers and then crosses out all even numbers, all numbers divisible by 3, and so on until reaching some upper limit – and whatever is left must be 1 or prime. Although there has been much research on the distribution of prime numbers within all the positive whole numbers during the past two centuries, for many purposes Eratosthenes' sieve is still the best methods available. A picture of this sieve for integers up to 100 appears on page 179 of Burton. Here is a link to a larger sieve going up to 400; this one is interactive and one can actually see the workings by clicking on the primes up to 19 in succession.

http://www.faust.fr.bw.schule.de/mhb/eratosiv.htm

Here are some links for important results on the distribution of primes:

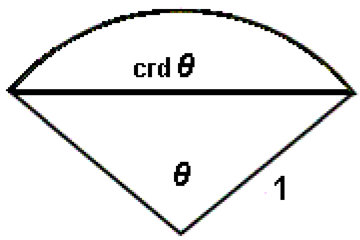
http://mathworld.wolfram.com/PrimeCountingFunction.html

http://mathworld.wolfram.com/PrimeNumberTheorem.html

Trigonometry and spherical geometry in Greek mathematics

We have already mentioned the increasingly prominent role of trigonometric studies in Greek mathematics and the links to astronomy. Many individuals who contributed to one of these fields also contributed to the other, and a great deal of work was done to tabulate values of trigonometrically related functions. Two particularly important names in this respect are Hipparchus of Rhodes (190 – 120 B.C.E.), to whom we owe concepts of latitude and longitude (and possibly the 360 degree circle), and Claudius Ptolemy (85 – 165 A.D.), whose *Almagest* was the definitive reference for astronomers until later in the sixteenth century. Incidentally, Claudius Ptolemy was not a king from the Ptolemaic dynasty that ruled Egypt during the time between the death of Alexander the Great and the conquest of Egypt by Octavian around 30 B.C.E. (with the defeat of Mark Anthony and Cleopatra),

Thanks to the work of individuals like Hipparchus and Ptolemy, Greek mathematicians constructed extensive tables of the chord function **crd,** whose value at an angle θ is the length of a chord in a circle of radius 1 that intercepts an arc whose angular measure is θ.



Of course, today we usually do not have separate tables for **crd** θ but find it by observing that **crd** θ is just twice the sine of $\frac{1}{2}\theta$.

Given the role of trigonometry in astronomical observations, one should more or less expect that Greek mathematicians were acquainted with many aspects of spherical geometry. The work of Menelaus of Alexandria (AD 70 – 130) is particularly significant in this respect and summarizes the knowledge of spherical geometry in ancient Greek mathematics. There is an extensive body of results in spherical geometry and trigonometry that has remarkable similarities to plane geometry in some respects but remarkable differences in others. On the surface of a sphere, the shortest distance between two points is along a great circle arc (*i.e.*, a circle whose center is also the center of the sphere), and accordingly spherical triangles are formed using three great circle arcs. There are congruence theorems for such triangles that are analogous to the standard congruence theorems for plane triangles, there are analogs of results like the Law of Sines and the Law of Cosines, but there is also an Angle-Angle-Angle congruence theorem for spherical triangles. At first this may seem surprising, but it reflects two important ways in which spherical triangles differ from plane triangles. The sums of the measures of the vertex angles are always greater than 180°, and in fact their areas are proportional to the excess of the angle sum over 180°; thus if the three angles have equal measures, then the triangles have the same area, and since we know that two triangles in plane geometry with equal angles and equal areas are congruent, the Angle-Angle-Angle congruence theorem should not seem all that shocking.

One other widely recognized name from that period is Heron (or Hero) of Alexandria (10 – 75 A.D.). Today he is best known for the formula giving the area of a triangle in terms of the lengths of its sides

$$\textbf{AREA} = \textbf{sqrt}\,\textbf{(}\; s\,(s-a)\,(s-b)\,(s-c)\;\textbf{)}$$

where a, b, c are the lengths of the sides and the semiperimeter s is equal to the familiar expression ½ (a + b + c). This result appears in several of his books with a derivation in his *Metrica*. There are statements (particularly in Arab commentaries) that this result had been known to earlier mathematicians including Archimedes, but Heron's proof is the earliest one that has survived. Heron's interests were extremely wide ranging, and he was particularly adept at applications of mathematical ideas to other areas including mechanics and geodesy. His analysis of reflected light very closely anticipated Fermat's minimum principle in optics 1500 years later. One reflection of Heron's interests in other subjects is how he mixed approximate and actual results to a degree one rarely finds in Greek mathematics.