4.A. Modern proofs of some ancient Greek results

The purpose of this document is to give proofs of some results due to Archimedes and Apollonius using modern methods from coordinate geometry and calculus.

Quadrature of the parabola

We shall approach this problem using the figure in the main notes. Suppose we consider the parabola $y = x^2$ and the region bounded by that curve that the line joining the two points $A = (-a, a^2)$ and $B = (b, b^2)$ on that parabola. If $C = (b, 0)$ and $D = (-a, 0)$, then A, B, C, D are the vertices of a trapezoid or parallelogram, two of whose sides are parallel to the $x - axis$ and one of whose sides lies on the y-axis. The area bounded by the parabola and the line is then the area bounded by the quadrilateral minus the area under the curve $y = x^2$. Elementary geometry implies that the area of the former is $\frac{1}{2}(a + b)(a^2 + b^2)$, and calculus imples that the area under the parabola is $\frac{1}{3}(a^3 + b^3)$. Subtracting the first from the second and simplifying, we find that the area bounded by the parabola and the line is equal to $\frac{1}{6}(a + b)^3$.

To find the area of the triangle, the first problem is to locate the point $E = (c, c^2)$ on the parabola such that the tangent line to the parabola at E is parallel to the chord joining A to B. In other words, we need to solve the following equation:

$$2\,c \quad = \quad \frac{b^2 - a^2}{b + a}$$

The answer is $\frac{1}{2}(b - a)$. Note that this number is equidistant from b and $-a$, and the common distance is equal to $\frac{1}{2}(a + b)$. Let F be the midpoint of segment $[AB]$, so that $F = (\frac{1}{2}(a + b),\, 0)$.

Archimedes' result compares the area of the parabolic sector with that of triangle ABE. This in turn is given by

$$\text{area } ABCD \; - \; (\,\text{area } ADFE \; + \; \text{area } BCFE\,)$$

where the areas are given as follows:

$$\text{area } ABCD \quad = \quad \frac{1}{2}(a + b)(a^2 + b^2)$$

$$\text{area } ADFE \quad = \quad \frac{1}{4}(a + b)\left(a^2 + \tfrac{1}{2}(b - a)^2\right)$$

$$\text{area } BDFE \quad = \quad \frac{1}{4}(a + b)\left(b^2 + \tfrac{1}{2}(b - a)^2\right)$$

If we work out the algebra and simplify, we find that the area of $\Delta\, ABE$ is $\frac{1}{8}(a + b)^3$, which is $\frac{3}{4}$ times the area of the parabolic sector.■

Volume of a paraboloid of revolution

Given a graph defined by $y = f(x) > 0$, the volume of the solid of revolution formed by rotating the region under this curve about the x-axis is equal to

$$\pi \cdot \int_a^b \; [f(x)]^2 \, dx \; .$$

If we begin with the parabola whose equation is $y^2 = 4\,p\,x$ and take the plane $x = h$ as the other piece of the solid region's boundary, we obtain the following integral:

$$\pi \cdot \int_0^h 4\,p\,x\,dx \quad = \quad 2\,p\,\pi\,h^2$$

On the other hand, the inscribed cone is the solid of revolution corresponding to the graph

$$y \quad = \quad 2\,x\,\sqrt{\frac{p}{h}}$$

and if we substitute this into the general formula we obtain an answer of $4\,p\,\pi\,h^2/3$. Since $2 = \frac{4}{3} \times \frac{3}{2}$, one retrieves Archimedes' formula relating the volume of the region (partly) bounded by the paraboloid of revolution and the inscribed cone.■

A result on tangents to a parabola

The following result is Proposition I–33 in Apollonius' books *On Conics*:

Theorem. *Let* **P** *be a parabola, let* **L** *be its axis of symmetry, suppose that E is the vertex of the parabola where these two curves meet. Given a point X on the parabola, let B be the foot of the perpendicular from X to* **L**, *and choose A so that E is the midpoint of $[AB]$. Then the line XA is the tangent line to* **L** *at X.*

Derivation using analytic geometry. Suppose that the parabola is $y = x^2$ (hence its axis of symmetry is the y-axis with equation $x = 0$), and that the point X has coordinates (c, c^2). Then the vertex E is the origin and B has coordinates $(0, c^2)$. Furthermore, the tangent line at X is defined by the following linear equation:

$$\frac{y - c^2}{x - c} \quad = \quad 2\,c$$

The theorem amounts to saying that this line passes through the point $A = (0, -c^2)$, so we need to check that A lies on the tangent line. This follows by direct substution of these coordinates for x and y in the equation of the line.■